Exercise 1. ***The WKB Method***

In continuum mechanics one deals with differential equations every day. A few of them can be solved exactly, but the majority can be solved only numerically or approximately. The WKB method is a very powerful method to find such approximate solutions of linear differential equations.

a) Let us start with the following easy, in general not solvable, differential equation

$$\epsilon^2 y''(x) - q(x)y(x) = 0 \tag{1}$$

with a smooth function $q(x)$ and explore, how the WKB method works.

(i) As a warm-up we assume that $q(x) = q_0$ is constant. Solve the differential equation by using an exponential ansatz and find the most general solution!

(ii) The hypothesis made in the WKB method is that an exponential ansatz is still a good ansatz for an approximate solution if ϵ is small. All that is necessary is to ensure the expansion is general enough. To be more practical, let us use the ansatz

$$y(x) = Ae^{\theta(x)/\epsilon^\alpha}\left(y_0(x) + \epsilon^\alpha y_1(x) + \dots\right) \tag{2}$$

where $\theta(x), y_0(x), y_1(x), \dots$ and α need to be determined. Insert this ansatz in Eq. (1) and find $\theta(x), y_0(x)$ and α in order to show that

$$y(x) \approx \frac{1}{\sqrt[4]{q(x)}}\left[a\exp\left(-\frac{1}{\epsilon}\int^x ds\sqrt{q(s)}\right) + b\exp\left(\frac{1}{\epsilon}\int^x ds\sqrt{q(s)}\right)\right] \tag{3}$$

Hint: After you inserted the ansatz in the equation and calculated the derivatives, choose α properly such that the terms are balanced. Then solve the equation in in every order of ϵ, i.e.

$$A_0\epsilon^0 + A_1\epsilon^1 + A_2\epsilon^2 + \dots = 0 \qquad \Rightarrow \quad 0 = A_0 = A_1 = A_2 = \dots \tag{4}$$

b) Now, we want to use the WKB method in order to solve the Webster horn equation. For flow of a gas in a long, thin duct, the equation for the velocity potential $\phi(x,t)$ $(v = \nabla\phi)$ is given by the Webster horn equation,

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial \log A}{\partial x}\frac{\partial\phi}{\partial x} = 0 \tag{5}$$

where $A(x)$ is the cross-sectional area of the duct. This differential equation can be derived from the fluid equations by considering small variations from the equilibrium state. It is assumed that the system is quasi one dimensional, the other two dimensions are covered in the cross-sectional area $A(x)$.

(i) Introduce the dimensionless coordinates $x' = x/L$ where L is the characteristic length over which $\log A$ varies and make the ansatz

$$\phi(x,t) = \phi(x)e^{-i\omega t}. \tag{6}$$

Show then that Eq. (5) can be written as

$$\left(\epsilon^2 \partial_{x'}^2 + 1\right) \tilde{\phi}(x') + \epsilon^2 \frac{\partial \tilde{A}}{\partial x'} \frac{\partial \tilde{\phi}}{\partial x'} = 0 \tag{7}$$

where $\tilde{f}(x') = f(x'L)$ and $\epsilon = \Lambda/L$ and $1/\Lambda = 2\pi/\lambda = k = \omega/c$.

(ii) In order to keep notation simple we replace $\tilde{\phi} \to \phi, x' \to x$ etc. Make the following ansatz

$$\phi(x) = e^{i\theta(x)/\epsilon}(\phi_0(x) + \epsilon\phi_1(x) + \dots) \tag{8}$$

and solve the differential equation in lowest order of ϵ.

(iii) Go back to the old notation used in Eq. (5) and find the general solution for $\phi(x,t)$! Which condition must be fulfilled such that the approximate solution is valid? How can the spatially dependent amplitude be explained? Discuss!

Exercise 2. ***Waves - Advanced***

The WKB method can also be applied to higher dimensional systems. To illustrate this we consider the wave equation with a spatially dependent speed velocity $c(\vec{x})$, i.e.

$$\Delta\phi - \frac{1}{c^2(\vec{x})} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{9}$$

where ϕ is the velocity potential, $v = \nabla\phi$. Again, the wave equation can be derived by considering small variation from the equilibrium state and linearizing the equation. Assume that $c(\vec{x}) = c_0\, g(\vec{x})$ and use the ansatz

$$\phi(\vec{x}, t) = (\phi_0 + \Lambda\phi_1 + \dots)e^{ik\theta(\vec{x}) - i\omega t}, \quad \Lambda = \frac{1}{k} = \frac{c_0}{\omega}, \tag{10}$$

where Λ is again much shorter than the typical length scale of $c(\vec{x})$.

a) Follow the procedure of the WKB method and derive the Eikonal equation

$$|\nabla\theta(\vec{x})|^2 = \frac{1}{g^2(\vec{x})} \tag{11}$$

as well as the transport equation

$$2\nabla\theta \cdot \nabla\phi_0 + \Delta\theta \cdot \phi_0 = 0 \tag{12}$$

What is the physical meaning of $\nabla\theta(\vec{x})$? Solving the transport equation is still a tough exercise but the solution for ϕ_0 can be estimated geometrically, how?

Hint: Perform a Taylor series of $\theta(\vec{x})$ in the ansatz and interpret the terms! Try to rewrite the transport equation as $\nabla(...) = 0$ and use the Gauss theorem with an appropriate surface!

b) Consider the following local variation of the speed velocity (we consider a two dimensional system, $\vec{x} = (x, y)$)

$$\frac{1}{g^2(\vec{x})} = \frac{1}{1 - \frac{\delta}{(1+(x/a)^2)(1+(y/b)^2)}} \approx 1 + \frac{\delta}{(1 + (x/a)^2)(1 + (y/b)^2)} \tag{13}$$

where we assume that the variation is small ($|\delta| \ll 1$). At $x = -\infty$ we have a plane wave moving in the $\vec{e}_x$-direction ($\theta(\vec{x}) = \vec{x} \cdot \vec{e}_x$). Linearize the Eikonal equation and find $\theta(\vec{x})$! Make a sketch of $\nabla\theta(\vec{x})$!

Hint: Write $\theta(\vec{x}) = \theta_0(\vec{x}) + \theta_1(\vec{x})$ where θ_1 is the contribution coming from the variation of the speed velocity. Insert this into the Eikonal equation and keep terms up to first order in θ_1.

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