Exercise 1. ***Virial theorem with stress***

In this task we want to generalize the virial theorem for the case with non-zero stress field $\overline{\overline{\sigma}}$. In that case the linear momentum equation applies

$$\rho \frac{\mathrm{d}\overline{v}}{\mathrm{d}t} = \rho \overline{g} + \overline{\nabla} \cdot \overline{\overline{\sigma}} \ . \tag{1}$$

We start by introducing the scalar moment of inertia for a continuous mass distribution

$$I = \int_V \mathrm{d}V \ \rho(\overline{r}) \left| \overline{r} \right|^2 \ , \tag{2}$$

and the scalar virial

$$G = \int_v \mathrm{d}V \ \rho(\overline{r}) \, \overline{g}(\overline{r}) \cdot \overline{r} \quad . \tag{3}$$

Take the 2$^{\mathrm{nd}}$ (total) time derivative of I, identify the kinetic energy K and the virial G in the expression and get the final result

$$\frac{1}{2} \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + G - \int_V \mathrm{d}V \ \mathrm{Tr} \left\{ \overline{\overline{\sigma}} \right\} + \int_{\partial V} \mathrm{d}S \, \overline{\overline{\sigma}} \cdot \overline{n} \cdot \overline{r} \quad . \tag{4}$$

Hint: use the divergence theorem

$$\int_V \mathrm{d}V \, \overline{\nabla} \cdot \overline{F} = \int_{\partial V} \mathrm{d}S \, \overline{F} \cdot \overline{n} \tag{5}$$

to rewrite the term $\int_V \mathrm{d}V \left(\overline{\nabla} \cdot \overline{\overline{\sigma}} \right) \cdot \overline{r}$ into the 2 last terms in Eq.(4).

Exercise 2. ***Equilibrium equations in spherical coordinates***

In equilibrium, the stress tensor of an elastic material within the linear regime has to satisfy the vector equation

$$\overline{\nabla} \cdot \overline{\overline{\sigma}} + \rho \overline{g} = \overline{0} \ . \tag{6}$$

Although it is undoubtedly the most elegant way to use the coordinate-independent description whenever it is possible, in practice one often has to stick to some set of coordinates. It is usually convenient to choose coordinates which in a sense "match" the problem. In this task you are encouraged to write the equation Eq.(6) in components using the spherical coordinates (r, θ, ϕ).

You may follow the derivation of the equilibrium equations using the cylindric coordinates in the lecture notes. Recall the general formula

$$\overline{\nabla} \cdot \overline{\overline{\sigma}} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q_1} \left(h_2 h_3 \overline{T}_1 \right) + \frac{\partial}{\partial q_2} \left(h_1 h_3 \overline{T}_2 \right) + \frac{\partial}{\partial q_3} \left(h_1 h_2 \overline{T}_3 \right) \right) \ , \tag{7}$$

with stress field $\overline{T}_i = \sum_j \sigma_{ij} \overline{e}_j$ (which we use only as an auxiliary quantity in this derivation). Further recall that

$$h_i = \sqrt{g_{ii}} \ , \tag{8}$$

where $\overline{\overline{g}}$ is the metric tensor. For the spherical coordinates this has a diagonal form diag $\left(1, r^2, r^2 \sin^2 \theta\right)$, thus

$$h_1 = 1 \ , \qquad h_2 = r \ , \qquad h_3 = r \sin\theta \ . \tag{9}$$

The covariant derivatives of the orthonormal basis in spherical coordinates are given by[1]

$$\frac{\partial}{\partial\theta}\overline{e}_r = \overline{e}_\theta \ , \qquad \frac{\partial}{\partial\theta}\overline{e}_\theta = -\overline{e}_r \ , \qquad \frac{\partial}{\partial\phi}\overline{e}_\phi = -\sin\theta\,\overline{e}_r - \cos\theta\,\overline{e}_\theta \ , \tag{10}$$

$$\frac{\partial}{\partial\phi}\overline{e}_r = \sin\theta\,\overline{e}_\phi \ , \qquad \frac{\partial}{\partial\phi}\overline{e}_\theta = \cos\theta\,\overline{e}_\phi \ . \tag{11}$$

The derivation is straightforward: do the derivatives in Eq.(7) and collect terms for each basis vector separately. Use the symmetry $\sigma_{ij} = \sigma_{ji}$ to simplify the results. You shall get

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{r\phi}}{\partial\phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta) + \rho g_r = 0 \ , \tag{12}$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\phi}}{\partial\phi} + \frac{1}{r}[(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}] + \rho g_\theta = 0 \ , \tag{13}$$

$$\frac{\partial\sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{1}{r}(2\sigma_{\theta\phi}\cot\theta + 3\sigma_{r\phi}) + \rho g_\phi = 0 \ . \tag{14}$$

Exercise 3. ***Equilibrium of a planet/star***

In this task we will use the above set of equations. We assume that the planet or star is spherically symmetric, thus any quantity may depend only on the radius r. Assume that the external pressure at the outer radius R of the object is p_{ext}.

3.1. Show that a solution with pure isotropic stress exists. Write the equation which has to be satisfied. Note that the density is in general a function of the strain (depends on compression); in this case you may treat it as a function of the position.

3.2. Use the equation derived in the previous subtask to recover the virial theorem from the first task in its special form.

[1] We do not write those which are zero. To verify the equations you may either use intuitive geometric approach or you may express the unit vectors in cartesian coordinates, perform the derivatives in cartesian coordinates and then reexpress the result using the spherical basis.