Exercise 1. ***Virial Equilibrium***

Consider a non-rotating, non-moving star in virial equilibrium. Assume that the star is a perfect sphere at some temperature T with radius R.

(a) Write down the virial equilibrium condition and the total energy of the star.

(b) What happens to the temperature and radius as energy is radiated away?

Exercise 2. ***Deformation and Strain***

From the lecture, recall that the deformation gradient tensor can be split into the symmetric strain tensor ϵ_{ij} and the antisymmetric rotation tensor ω_{ij}. In particular, $G_{ij} = \epsilon_{ij} + \omega_{ij}$, where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

Here, $u_i = (u_1, u_2, u_3)$ is the displacement field, and $x_i = (x_1, x_2, x_3)$ the coordinate vector.

(a) Consider the displacement field $u_i = (u_x, u_y, u_z) = (a, 0, 0)$. Compute ϵ_{ij} and ω_{ij}. Interpret the result. Work in Cartesian coordinates.

(b) Now consider the displacement field $u_i = (u_x, u_y, u_z) = (cy, -cx, 0)$, where c is an arbitrary constant. Again, compute ϵ_{ij} and ω_{ij}. Interpret the result. Work in Cartesian coordinates.

(c) Consider the displacement field $u_i = (u_r, u_\theta, u_\phi) = (-pr, 0, 0)$, where p is an arbitrary constant. Compute the strain tensor ϵ_{ij}. Interpret the results. Work in spherical coordinates.

(d) Finally, consider the displacement field $u_i = (u_r, u_\theta, u_z) = (0, \alpha rz, 0)$, where α is some constant. Compute the strain tensor ϵ_{ij}. Interpret the results. Work in cylindrical coordinates.

Exercise 3. ***Maximum Height of a Mountain***

The equilibrium condition for the stress tensor is, in vector and index notation, respectively

$$\vec{\nabla} \cdot \bar{\bar{\sigma}} + \vec{F} = 0, \qquad \frac{\partial \sigma_{ij}}{\partial x_i} + F_j = 0, \tag{2}$$

where $x_i = (x, y, z)$, and $F_j = (F_x, F_y, F_z)$.

(a) Assume no shear stresses, an isotropic stress tensor, and a force with only a z-dependent z-component, i.e. $F_j = (F_x, F_y, F_z) = (0, 0, f(z))$. Show that the non-zero components of the stress tensor depend only on z.

(b) Use your previous findings to compute the maximum height of a mountain on (i) Earth, and (ii) Mars. Assume that the mountain is made out of granite. Compare with reality. Interpret any differences you might find.

(c) As (b), but assume a mountain made of iron. Only consider Earth. Interpret the result.

(d) A crude way to define a planet could be to impose the maximum height h of a mountain on the surface to be smaller than the radius. If the mountains are larger than the radius, we are dealing with an irregularly shaped object – an asteroid. With this in mind, find some data on a random asteroid of your choice, and compute h. Assume it is made out of granite. Is h comparable to the radius/size of the asteroid?