Superstrings

ETH Proseminar in Theoretical Physics

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May 13th, 2013

Outline

Outline

NSR action

Light cone gauge fixing

The Neveu–Schwarz–Ramond action in light cone gauge

$$S^{l.c.}_{NSR} = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X^i \partial^\alpha X^i - i\bar{\psi}^i \rho^\alpha \partial_\alpha \psi^i\right)$$

transverse coordinates $i = 1, \ldots, 8$

- ψ^i is a worldsheet Majorana 2-spinor
- ψ^i is a spacetime vector
- SO(8) rotational symmetry $\implies$ ψ^i is an $\mathbf{8_v}$ representation of SO(8)

Building up the spectrum

Tools for generating the spectrum

Open string or the **right-moving** part of a closed string.
Creation and annihilation operators:

- X^i : α^i_{-n} and α^i_n $\quad n \in \mathbb{Z}$
- ψ^i : $\begin{cases} d^i_{-m} \text{ and } d^i_m & m \in \mathbb{Z} & \text{for R-sector} \\ b^i_{-r} \text{ and } b^i_r & r \in \mathbb{Z} + \frac{1}{2} & \text{for NS-sector} \end{cases}$

The mass^2 operator:

$$\alpha' m^2 = \underbrace{\sum_{n>0} \alpha^i_{-n}\alpha^i_n}_{N^{(\alpha)}} + \underbrace{\sum_{m>0} m d^i_{-m} d^i_m}_{N^{(d)}} - \frac{1}{2} \qquad \text{for NS-sector}$$

$$\alpha' m^2 = \underbrace{\sum_{n>0} \alpha^i_{-n}\alpha^i_n}_{N^{(\alpha)}} + \underbrace{\sum_{r>0} r b^i_{-r} b^i_r}_{N^{(b)}} \qquad \text{for R-sector}$$

Building up the spectrum

Ground states

NSR formulation: All creation/annihilation operators are spacetime vectors.

- Bosons $\longrightarrow$ bosons
- Fermions $\longrightarrow$ fermions
- Ground state determines the type of spectrum (bosonic or fermionic) built on it

- NS-sector (spacetime bosons)
  - Scalar ground state (bosonic): $|0\rangle$
  - Tachyonic: $\alpha' m^2 = -\frac{1}{2}$
- R-sector (spacetime fermions)
  - $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \implies$ spinor of SO(1,9)
  - Spinor ground state (fermionic): $|c\rangle\, \chi^c(k)$
    $\chi^c(k)$ is a spinor, k is the momentum
  - Massless: $\alpha' m^2 = 0$
  - $D = 10 \implies 2^{D/2} = 32$ complex components: $c = 1, \ldots, 32$

Constraints on the R ground state

Impose two constraints *simultaneously* on the R ground state!

1. Majorana constraint
2. Weyl constraint

Also, ground state spinor satisfies the Dirac equation.

3. Dirac equation

Each condition will reduce the degrees of freedom from the original 64 (32 complex) by a factor of two:

$$\text{32 complex} \xrightarrow{\text{Majorana}} \text{32 real} \xrightarrow{\text{Weyl}} \text{16 real} \xrightarrow{\text{Dirac equation}} \text{8 real}$$

Crucial for spacetime supersymmetry!

Constraints on the R ground state

Majorana constraint

1. Majorana constraint: reality condition on the spinor field
   - The massless Dirac equation:

$$i\Gamma^{\mu}\partial_{\mu}\chi = 0$$

   Γ^{μ} – 10 generally complex 32 dimensional Dirac matrices.

   - If all Dirac matrices are **real** or **imaginary** ("Majorana representation"), then possible to impose reality on χ. ("Majorana spinor")
   - This is possible in $D = 2, 3, 4 \pmod 8$.
   - Construct **imaginary** Dirac matrices just for $D = 10$!

SO(8) Clifford algebra

- Spinor representation of SO(8): $\lambda = (\lambda_s^a, \lambda_c^{\bar{a}})$

  Reducible $\mathbf{8_s} \oplus \mathbf{8_c}$ representation $\quad a = 1, \ldots, 8$ *spinor index*; $\bar{a} = 1, \ldots, 8$ *conjugate spinor index*

- The SO(8) transformations can be constructed with the help of the Dirac matrices, which obey the Clifford algebra.

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}$$

$$\gamma^i = \begin{pmatrix} 0 & \gamma^i_{a\bar{a}} \\ \gamma^i_{\bar{a}a} & 0 \end{pmatrix}_{16\times 16} \qquad i, j = 1, \ldots, 8 \quad \textit{vector index}$$

- **Real, symmetric** $\gamma^i_{a\bar{a}}$ matrices constructed from the Pauli matrices:

$$\gamma^1_{a\bar{a}} = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 \qquad \gamma^2_{a\bar{a}} = \mathbb{1} \otimes \sigma_1 \otimes i\sigma_2$$

$$\gamma^3_{a\bar{a}} = \mathbb{1} \otimes \sigma_3 \otimes i\sigma_2 \qquad \gamma^4_{a\bar{a}} = \sigma_1 \otimes i\sigma_2 \otimes \mathbb{1}$$

$$\gamma^5_{a\bar{a}} = \sigma_3 \otimes i\sigma_2 \otimes \mathbb{1} \qquad \gamma^6_{a\bar{a}} = i\sigma_2 \otimes \mathbb{1} \otimes \sigma_1$$

$$\gamma^7_{a\bar{a}} = i\sigma_2 \otimes \mathbb{1} \otimes \sigma_3 \qquad \gamma^8_{a\bar{a}} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$$

SO(1,9) Clifford algebra

- The ($2^5 = 32$)-dimensional Dirac matrices of the SO(1,9) spinor representation obey the Clifford algebra as well:

$$\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu} \qquad \mu, \nu = 0, \dots, 9$$

- SO(8) is the transverse subgroup of SO(1,9). Use γ^i matrices to build the Γ^μ matrices

$$\begin{aligned} \Gamma^0 &= \sigma_2 \otimes \mathbb{1}_{16} \\ \Gamma^i &= i\sigma_1 \otimes \gamma^i \qquad i = 1, \dots, 8 \\ \Gamma^9 &= i\sigma_3 \otimes \mathbb{1}_{16} \end{aligned}$$

- γ^i are all **real** $\implies$ Γ^μ are all **imaginary**
  The Majorana representation is possible!

$$\chi\text{: 32 complex} \longrightarrow \text{32 real components}$$

Constraints on the R ground state

Weyl constraint

2. Weyl constraint: the spinor field has definite chirality
   - Define chirality operator: $\Gamma_{11} = \Gamma^0\Gamma^1 \cdots \Gamma^9$
   - Anticommutes with all the other Dirac matrices and squares to identity:
   $$\{\Gamma_{11}, \Gamma^\mu\} = 0 \qquad (\Gamma_{11})^2 = 1$$
   - Spinor of definite chirality ("Weyl spinor"):
   $$\Gamma_{11}\chi = \pm\chi$$
   - By demanding positive or negative chirality, again eliminate half of the degrees of freedom
   $$\begin{array}{ccccc} \chi : \mathbf{16_s} \oplus \mathbf{16_c} & \longrightarrow & \lambda : \mathbf{8_s} \oplus \mathbf{8_c} & & \\ 32 \text{ real} & \longrightarrow & 16 \text{ real} & & \text{components} \end{array}$$
   - Note: Majorana and Weyl constraints are only compatible in $D = 2 \pmod 8$ dimensions.

Constraints on the R ground state

Dirac equation

3 Dirac equation (massless):

$$i\,\Gamma^\mu \partial_\mu \chi = 0$$

- For Weyl spinors $\lambda = (\lambda_s, \lambda_c)$, this reduces to

$$(\partial_0 \pm \partial_9)\,\lambda_s^a + \gamma^i_{a\bar{a}} \partial_i \lambda_c^{\bar{a}} = 0 \qquad a = 1, \ldots, 8$$
$$(\partial_0 \mp \partial_9)\,\lambda_c^{\bar{a}} + \gamma^i_{\bar{a}a} \partial_i \lambda_s^a = 0 \qquad \bar{a} = 1, \ldots, 8$$

  for chirality $\Gamma_{11}\chi = \pm\chi$
- The spinor and the conjugate spinor representations are not independent for a definite chirality!

$$\begin{array}{ccc} \lambda : \mathbf{8_s} \oplus \mathbf{8_c} & \longrightarrow & \lambda_s : \mathbf{8_s} \quad \text{or} \quad \lambda_c : \mathbf{8_c} \\ \text{16 real} & \longrightarrow & \text{8 real components} \end{array}$$

- $$\left.\begin{array}{l} + \text{ chirality} \to \text{Choose } \mathbf{8_s} \\ - \text{ chirality} \to \text{Choose } \mathbf{8_c} \end{array}\right\} \text{form } \mathbf{8_s} \oplus \mathbf{8_c} \text{ multiplet}$$

Open string spectrum

Without GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	little group	representation contents with respect to the little group
	NS-sector (bosons)		
$-\frac{1}{2}$	$\|0\rangle$ $\mathbf{1}$	SO(9)	$\mathbf{1}$
0	$b^i_{-1/2}\|0\rangle$ $\mathbf{8_v}$	SO(8)	$\mathbf{8_v}$
$+\frac{1}{2}$	$\alpha^i_{-1}\|0\rangle$ $\mathbf{8_v}$; $b^i_{-1/2}b^j_{-1/2}\|0\rangle$ $\mathbf{28}$	SO(9)	$\mathbf{36}$
$+1$	$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\|0\rangle$ $\mathbf{56_v}$ $\alpha^i_{-1}b^j_{-1/2}\|0\rangle$ $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}$; $b^i_{-3/2}\|0\rangle$ $\mathbf{8_v}$	SO(9)	$\mathbf{84}\oplus\mathbf{44}$
	R-sector (fermions)		
0	$\|a\rangle$ $\mathbf{8_s}$ $\|\bar{a}\rangle$ $\mathbf{8_c}$	SO(8)	$\mathbf{8_s}$ $\mathbf{8_c}$
$+1$	$\alpha^i_{-1}\|a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$; $d^i_{-1}\|\bar{a}\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$ $\alpha^i_{-1}\|\bar{a}\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$; $d^i_{-1}\|a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$	SO(9)	$\mathbf{128}$ $\mathbf{128}$

$i, j, k = 1 \ldots, 8 \qquad a, \bar{a} = 1, \ldots, 8$

GSO projection

Why are we not satisfied?

Several properties of the NSR model so far which are not so tempting:

- Tachyon
- Anticommuting operators map bosons to bosons
- No spacetime supersymmetry:
  $$\#\text{bosons} \neq \#\text{fermions on the same mass levels}$$
- Conflict with modular invariance

Gliozzi, Scherk and Olive (GSO): **truncate the spectrum**

Will seem arbitrary at first, but the GSO projection is required by modular invariance.

GSO projection

Truncation of the NS-sector

Let $|\varphi_0\rangle$ be a bosonic state.

$$|\varphi\rangle = b^{i_1}_{-r_1} b^{i_2}_{-r_2} \cdots b^{i_n}_{-r_n} |\varphi_0\rangle$$

$b^i_{-r_i}$ are spacetime vectors $\implies$ for $\forall n$, $|\varphi\rangle$ is bosonic.

- **n is even:** product of the anticommuting operators is *commuting* *commuting* operator: bosons $\longrightarrow$ bosons
- **n is odd:** product of the anticommuting operators is *anticommuting* *anticommuting* operator: bosons $\longrightarrow$ bosons

GSO projection: discard those with n odd.
Still have a choice in $|\varphi_0\rangle$ states of reference. Appealing requirements:

- Get rid of the tachyon!
- #bosons = #fermions on the same mass levels!

Formally:

- Define quantum number: $G = -(-1)^F$ where $F = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r$
- Demand: $G|\varphi\rangle = +|\varphi\rangle$

GSO projection

Truncation of the R-sector

NS-sector truncation: all mass levels exist in both the fermionic and the bosonic sector, but still #bosons $\neq$ #fermions on the same mass levels.
GSO projection: eliminate half of the R-sector!
Formally:

- Generalize the chirality operator Γ_{11} for massive levels:

$$\bar{\Gamma} = \Gamma_{11} \left(-1\right)^{F} \qquad \text{again } F = \sum_{m=1}^{\infty} d_{-m}^{i} d_{m}^{i}$$

- Demand: $\bar{\Gamma} \left|\chi\right\rangle = \pm \left|\chi\right\rangle$
- $\{ \bar{\Gamma} \, , d_{n}^{\mu} \} = 0 \implies$ Projection depends on the ground state. For the ground state $\bar{\Gamma} = \Gamma_{11} \implies$ Only keep the states built onto the $+$ or $-$ chirality massless Weyl spinors.
- *Note:* This does not mean that the massive states are Weyl spinors. Massive spinors cannot be Weyl!

Open string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G (NS) $\bar{\Gamma}$ (R)	little group	representation contents with respect to the little group
	NS-sector (bosons)			
$-\frac{1}{2}$	$\|0\rangle$ **1**	-1	SO(9)	**1**
0	$b^i_{-1/2}\|0\rangle$ $\mathbf{8_v}$	$+1$	SO(8)	$\mathbf{8_v}$
$+\frac{1}{2}$	$\alpha^i_{-1}\|0\rangle$ $\mathbf{8_v}$; $b^i_{-1/2}b^j_{-1/2}\|0\rangle$ **28**	-1	SO(9)	**36**
$+1$	$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\|0\rangle$ $\mathbf{56_v}$; $\alpha^i_{-1}b^j_{-1/2}\|0\rangle$ $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}$; $b^i_{-3/2}\|0\rangle$ $\mathbf{8_v}$	$+1$	SO(9)	$\mathbf{84}\oplus\mathbf{44}$
	R-sector (fermions)			
0	$\|a\rangle$ $\mathbf{8_s}$	$+1$	SO(8)	$\mathbf{8_s}$
	$\|\bar{a}\rangle$ $\mathbf{8_c}$	-1		$\mathbf{8_c}$
$+1$	$\alpha^i_{-1}\|a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$; $d^i_{-1}\|\bar{a}\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$	$+1$	SO(9)	**128**
	$\alpha^i_{-1}\|\bar{a}\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$; $d^i_{-1}\|a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$	-1		**128**

$i, j, k = 1 \ldots, 8 \qquad a, \bar{a} = 1, \ldots, 8$

Open string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G (NS) $\bar{\Gamma}$ (R)	little group	representation contents with respect to the little group
	NS-sector (bosons)			
~~$-\frac{1}{2}$~~	~~$\|0\rangle$ $\mathbf{1}$~~	~~-1~~	~~SO([illegible])~~	~~$\mathbf{1}$~~
0	$b^i_{-1/2}\|0\rangle$ $\mathbf{8_v}$	$+1$	SO(8)	$\mathbf{8_v}$
~~$+\frac{1}{2}$~~	~~$\alpha^i_{-1}\|0\rangle$ $\quad$ $b^i_{-1/2}b^j_{-1/2}\|0\rangle$ $\mathbf{8_v}$ $\quad$ $\mathbf{28}$~~	~~-1~~	~~SO([illegible])~~	~~$\mathbf{36}$~~
$+1$	$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\|0\rangle$ $\mathbf{56_v}$ $\alpha^i_{-1}b^j_{-1/2}\|0\rangle$ $\quad$ $b^i_{-3/2}\|0\rangle$ $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}$ $\quad$ $\mathbf{8_v}$	$+1$	SO(9)	$\mathbf{84}\oplus\mathbf{44}$
	R-sector (fermions)			
0	$\|a\rangle$ $\mathbf{8_s}$ $\|\bar{a}\rangle$ $\mathbf{8_c}$	$+1$ -1	SO(8)	$\mathbf{8_s}$ $\mathbf{8_c}$
$+1$	$\alpha^i_{-1}\|a\rangle$ $\quad$ $d^i_{-1}\|\bar{a}\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$ $\quad$ $\mathbf{8_s}\oplus\mathbf{56_s}$ $\alpha^i_{-1}\|\bar{a}\rangle$ $\quad$ $d^i_{-1}\|a\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$ $\quad$ $\mathbf{8_c}\oplus\mathbf{56_c}$	$+1$ -1	SO(9)	$\mathbf{128}$ $\mathbf{128}$

$i, j, k = 1\ldots, 8 \qquad a, \bar{a} = 1, \ldots, 8$

Open string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G (NS) $\bar{\Gamma}$ (R)	little group	representation contents with respect to the little group
		NS-sector (bosons)		
~~$[illegible]\frac{1}{2}$~~	~~$\vert 0\rangle$ **1**~~	~~[illegible]~~	~~SO([illegible])~~	~~[illegible]~~
0	$b^i_{-1/2}\,\vert 0\rangle$ $\mathbf{8_v}$	$+1$	SO(8)	$\mathbf{8_v}$
~~$[illegible]\frac{1}{2}$~~	~~$\alpha^i_{[illegible]}\,\vert 0\rangle$ $b^i_{[illegible]} b^j_{[illegible]}\,\vert 0\rangle$ $\mathbf{8_v}$ **28**~~	~~[illegible]~~	~~SO([illegible])~~	~~[illegible]~~
$+1$	$b^i_{-1/2} b^j_{-1/2} b^k_{-1/2}\,\vert 0\rangle$ $\mathbf{56_v}$ $\alpha^i_{-1} b^j_{-1/2}\,\vert 0\rangle$ $b^i_{-3/2}\,\vert 0\rangle$ $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$ $\mathbf{8_v}$	$+1$	SO(9)	$\mathbf{84} \oplus \mathbf{44}$
		R-sector (fermions)		
0	$\vert a\rangle$ $\mathbf{8_s}$ ~~$\vert \bar{a}\rangle$ $\mathbf{8_c}$~~	$+1$ ~~[illegible]~~	SO(8)	$\mathbf{8_s}$ ~~[illegible]~~
$+1$	$\alpha^i_{-1}\,\vert a\rangle$ $d^i_{-1}\,\vert \bar{a}\rangle$ $\mathbf{8_c} \oplus \mathbf{56_c}$ $\mathbf{8_s} \oplus \mathbf{56_s}$ ~~$\alpha^i_{[illegible]}\,\vert \bar{a}\rangle$ $d^i_{[illegible]}\,\vert a\rangle$ $\mathbf{8_s} \oplus \mathbf{56_s}$ $\mathbf{8_c} \oplus \mathbf{56_c}$~~	$+1$ ~~[illegible]~~	SO(9)	**128** ~~[illegible]~~

$i, j, k = 1 \ldots, 8 \qquad a, \bar{a} = 1, \ldots, 8$

Open string spectrum
With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G (NS) $\bar{\Gamma}$ (R)	little group	representation contents with respect to the little group
NS-sector (bosons)				
~~$-\frac{1}{2}$~~	~~$\lvert 0\rangle$ **1**~~	~~-1~~	~~SO([illegible])~~	~~**1**~~
0	$b^i_{-1/2}\lvert 0\rangle$ $\mathbf{8_v}$	$+1$	SO(8)	$\mathbf{8_v}$
~~$+\frac{1}{2}$~~	~~$\alpha^i_{-1}\lvert 0\rangle$ $\mathbf{8_v}$; $b^i_{-1/2}b^j_{-1/2}\lvert 0\rangle$ **28**~~	~~-1~~	~~SO(9)~~	~~**36**~~
$+1$	$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\lvert 0\rangle$ $\mathbf{56_v}$; $\alpha^i_{-1}b^j_{-1/2}\lvert 0\rangle$ $\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35_v}$; $b^i_{-3/2}\lvert 0\rangle$ $\mathbf{8_v}$	$+1$	SO(9)	$\mathbf{84}\oplus\mathbf{44}$
R-sector (fermions)				
0	~~$\lvert a\rangle$ $\mathbf{8_s}$~~	~~$+1$~~	SO(8)	~~$\mathbf{8_s}$~~
	$\lvert \bar{a}\rangle$ $\mathbf{8_c}$	-1		$\mathbf{8_c}$
$+1$	~~$\alpha^i_{-1}\lvert a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$; $d^i_{-1}\lvert \bar{a}\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$~~	~~$+1$~~	SO(9)	~~**128**~~
	$\alpha^i_{-1}\lvert \bar{a}\rangle$ $\mathbf{8_s}\oplus\mathbf{56_s}$; $d^i_{-1}\lvert a\rangle$ $\mathbf{8_c}\oplus\mathbf{56_c}$	-1		**128**

$i, j, k = 1\ldots, 8 \qquad a, \bar{a} = 1, \ldots, 8$

Closed string spectrum

Tensoring together states

Closed string spectrum: tensor product of the left- and right-movers

Bosons: (NS,NS) and (R,R) Fermions: (NS,R) and (R,NS)

- $L_0 = \tilde{L}_0 \implies m_L^2 = m_R^2$
  $\implies$ only states of the same mass levels can be tensored together
- GSO projection: separately for left- and right-movers

- **type IIA** theory
  - NS-sector: $G_L = G_R = +1$
  - R-sector: $\bar{\Gamma}_L = -\bar{\Gamma}_R = 1$
- **type IIB** theory
  - NS-sector: $G_L = G_R = +1$
  - R-sector: $\bar{\Gamma}_L = \bar{\Gamma}_R = 1$

- Only the massless level is different!

Closed string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G_L (NS) $\bar{\Gamma}_L$ (R)	G_R (NS) $\bar{\Gamma}_R$ (R)	little group	representation contents with respect to the little group
	(NS,NS)-sector (bosons)				
-2	$\lvert 0\rangle_L \otimes \lvert 0\rangle_R$ $\mathbf{1} \otimes \mathbf{1}$	-1	-1	SO(9)	$\mathbf{1}$
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes b^j_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$
	(R,R)-sector (bosons)				
0	$\lvert a\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}$
	$\lvert \bar{a}\rangle_L \otimes \lvert \bar{b}\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_c}$	-1	-1		$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c}$
	$\lvert \bar{a}\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_s}$	-1	$+1$		$\mathbf{8_v} \oplus \mathbf{56_v}$
	$\lvert a\rangle_L \otimes \lvert \bar{b}\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_v} \oplus \mathbf{56_v}$
	(R,NS)-sector (fermions)				
0	$\lvert a\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	$\lvert \bar{a}\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_v}$	-1	$+1$		$\mathbf{8_s} \oplus \mathbf{56_s}$
	(NS,R)-sector (fermions)				
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert a\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert \bar{a}\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_s} \oplus \mathbf{56_s}$

Closed string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G_L (NS) $\bar{\Gamma}_L$ (R)	G_R (NS) $\bar{\Gamma}_R$ (R)	little group	representation contents with respect to the little group
	(NS,NS)-sector (bosons)				
-2	$\lvert 0\rangle_L \otimes \lvert 0\rangle_R$ $\mathbf{1} \otimes \mathbf{1}$	-1	-1	SO(9)	$\mathbf{1}$
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes b^j_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$
	(R,R)-sector (bosons)				
0	$\lvert a\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}$
	$\lvert \bar{a}\rangle_L \otimes \lvert \bar{b}\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_c}$	-1	-1		$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c}$
	$\lvert \bar{a}\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_s}$	-1	$+1$		$\mathbf{8_v} \oplus \mathbf{56_v}$
	$\lvert a\rangle_L \otimes \lvert \bar{b}\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_v} \oplus \mathbf{56_v}$
	(R,NS)-sector (fermions)				
0	$\lvert a\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	$\lvert \bar{a}\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_v}$	-1	$+1$		$\mathbf{8_s} \oplus \mathbf{56_s}$
	(NS,R)-sector (fermions)				
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert a\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert \bar{a}\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_s} \oplus \mathbf{56_s}$

Closed string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G_L (NS) $\bar{\Gamma}_L$ (R)	G_R (NS) $\bar{\Gamma}_R$ (R)	little group	representation contents with respect to the little group
(NS,NS)-sector (bosons)					
~~-2~~	~~$\lvert 0\rangle_L \otimes \lvert 0\rangle_R$ $\mathbf{1} \otimes \mathbf{1}$~~	~~-1~~	~~-1~~	~~SO(9)~~	~~$\mathbf{1}$~~
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes b^j_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$
(R,R)-sector (bosons)					
0	~~$\lvert a\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_s}$~~	~~$+1$~~	~~$+1$~~	SO(8)	~~$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}$~~
	~~$\lvert \bar{a}\rangle_L \otimes \lvert \bar{b}\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_c}$~~	~~-1~~	~~-1~~		~~$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c}$~~
	~~$\lvert \bar{a}\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_s}$~~	~~-1~~	~~$+1$~~		~~$\mathbf{8_v} \oplus \mathbf{56_v}$~~
	$\lvert a\rangle_L \otimes \lvert \bar{b}\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_v} \oplus \mathbf{56_v}$
(R,NS)-sector (fermions)					
0	$\lvert a\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	~~$\lvert \bar{a}\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_c} \otimes \mathbf{8_v}$~~	~~-1~~	~~$+1$~~		~~$\mathbf{8_s} \oplus \mathbf{56_s}$~~
(NS,R)-sector (fermions)					
0	~~$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert a\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_s}$~~	~~$+1$~~	~~$+1$~~	SO(8)	~~$\mathbf{8_c} \oplus \mathbf{56_c}$~~
	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert \bar{a}\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_c}$	$+1$	-1		$\mathbf{8_s} \oplus \mathbf{56_s}$

Closed string spectrum

With GSO projection

$\alpha' m^2$	states and their SO(8) representation contents	G_L (NS) $\bar{\Gamma}_L$ (R)	G_R (NS) $\bar{\Gamma}_R$ (R)	little group	representation contents with respect to the little group
	(NS,NS)-sector (bosons)				
~~-2~~	~~$\lvert 0\rangle_L \otimes \lvert 0\rangle_R$~~ ~~$\mathbf{1} \otimes \mathbf{1}$~~	~~-1~~	~~-1~~	~~SO(8)~~	~~$\mathbf{1}$~~
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes b^j_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$
	(R,R)-sector (bosons)				
0	$\lvert a\rangle_L \otimes \lvert b\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_s}$
	~~$\lvert \bar{a}\rangle_L \otimes \lvert \bar{b}\rangle_R$~~ ~~$\mathbf{8_c} \otimes \mathbf{8_c}$~~	~~-1~~	~~-1~~		~~$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c}$~~
	~~$\lvert \bar{a}\rangle_L \otimes \lvert b\rangle_R$~~ ~~$\mathbf{8_c} \otimes \mathbf{8_s}$~~	~~-1~~	~~$+1$~~		~~$\mathbf{8_v} \oplus \mathbf{56_v}$~~
	~~$\lvert a\rangle_L \otimes \lvert \bar{b}\rangle_R$~~ ~~$\mathbf{8_s} \otimes \mathbf{8_c}$~~	~~$+1$~~	~~-1~~		~~$\mathbf{8_v} \oplus \mathbf{56_v}$~~
	(R,NS)-sector (fermions)				
0	$\lvert a\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$ $\mathbf{8_s} \otimes \mathbf{8_v}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	~~$\lvert \bar{a}\rangle_L \otimes b^i_{-1/2} \lvert 0\rangle_R$~~ ~~$\mathbf{8_c} \otimes \mathbf{8_v}$~~	~~-1~~	~~$+1$~~		~~$\mathbf{8_s} \oplus \mathbf{56_s}$~~
	(NS,R)-sector (fermions)				
0	$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert a\rangle_R$ $\mathbf{8_v} \otimes \mathbf{8_s}$	$+1$	$+1$	SO(8)	$\mathbf{8_c} \oplus \mathbf{56_c}$
	~~$\tilde{b}^i_{-1/2} \lvert 0\rangle_L \otimes \lvert \bar{a}\rangle_R$~~ ~~$\mathbf{8_v} \otimes \mathbf{8_c}$~~	~~$+1$~~	~~-1~~		~~$\mathbf{8_s} \oplus \mathbf{56_s}$~~

Closed string spectrum

Massless spectrum of type II theories

Massless spectrum:

Type IIA Bosons:
$[\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v] \oplus [\mathbf{8}_v \oplus \mathbf{56}_v]$

Fermions:
$[\mathbf{8}_c \oplus \mathbf{56}_c] \oplus [\mathbf{8}_s \oplus \mathbf{56}_s]$

Type IIB Bosons:
$[\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v] \oplus [\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s]$

Fermions:
$[\mathbf{8}_c \oplus \mathbf{56}_c] \oplus [\mathbf{8}_c \oplus \mathbf{56}_c]$

- $\mathbf{1}$ – dilaton
- $\mathbf{28}$ – rank 2 antisymmetric tensor
- $\mathbf{35}_v$ – graviton
- $\mathbf{8}_v$ – vector
- $\mathbf{56}_v$ – rank 3 antisymmetric tensor
- $\mathbf{8}_{s/c}$ – dilatinos of opposite chirality
- $\mathbf{56}_{s/c}$ – gravitinos of opposite chirality
- $\mathbf{35}_s$ – rank 4 self-dual antisymmetric tensor

Outline

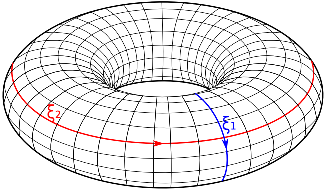
Spin structures

Investigate loop partition functions (vacuum bubbles):

- Supersymmetric theory: the loop partition functions vanish due to equal positive and negative contributions respectively from bosons and fermions.
- g-loop vacuum bubble worldsheet is conformally equivalent to a class of Riemann surfaces of genus g
- Riemann surface of genus g: $2g$ uncontractible loops.
- ψ-field: periodicity or antiperiodicity for each of the loops: 2^{2g} choices, 2^{2g} different **spin structures**
- Even/odd spin structures: Number of zero modes of the chiral Dirac operator (∇_z or $\nabla_{\bar{z}}$) is even/odd

Investigate only the one-loop partition function.

Spin structures
Torus



One-loop vacuum bubble worldsheet: conformally equivalent to a class of torii.

- Four possible spin structures corresponding to boundary conditions along the two loops: $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$
- Put globally flat metric onto torus: $\nabla_z \to \partial_z$
- Only global zero mode: constant spinor $\Longrightarrow$ $(+,+)$ boundary condition

$$(+,+) \to 1 \text{ zero mode: odd}$$
$$(+,-) \to 0 \text{ zero mode: even}$$
$$(-,+) \to 0 \text{ zero mode: even}$$
$$(-,-) \to 0 \text{ zero mode: even}$$

Spin structures

Higher genus Riemann surfaces

Two statements:

1. For a given spin structure, the number of chiral Dirac zero modes is a topological invariant modulo two.
2. The number of chiral Dirac zero modes is additive modulo two when gluing together two Riemann surfaces.

The number of even and odd spin structures on an arbitrary Riemann surface of genus g (from spin structure on torus and 2):

- Odd: $\sum_{m \text{ odd}} \binom{g}{m} 3^{g-m} = 2^{g-1}(2^g - 1)$
- Even: $\sum_{m \text{ even}} \binom{g}{m} 3^{g-m} = 2^{g-1}(2^g + 1)$

Spin structures

Back to the torus

Recall:

- Parametrize torus by $\xi^1, \xi^2 \in [0, 1]$
- Complex coordinates: $z = \xi^1 + \tau\xi^2, \quad \bar{z} = \xi^1 + \bar{\tau}\xi^2$
- Modular transformations of τ (Teichmüller parameter)

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

  lead to locally conformally equivalent tori.

- Generator of modular transformations:

$$S : \tau \longrightarrow -1/\tau \qquad \Longrightarrow \left(\xi^1, \xi^2\right) \longrightarrow \left(-\xi^2, \xi^1\right)$$

$$T : \tau \longrightarrow \tau + 1 \qquad \Longrightarrow \left(\xi^1, \xi^2\right) \longrightarrow \left(\xi^1 + \xi^2, \xi^2\right)$$

Spin structures

Boundary conditions under modular transformations

- Four possible boundary conditions (spin structures)

$$\psi(\xi^1,\xi^2) = \pm\psi(\xi^1+1,\xi^2) \qquad \text{(NS or R-sector)}$$
$$\psi(\xi^1,\xi^2) = \pm\psi(\xi^1,\xi^2+1)$$

- How do the boundary conditions transform under modular transformations?

$$S: \qquad (+,+) \longrightarrow (+,+), \qquad \begin{array}{c}(+,-) \longrightarrow (-,+)\\ (-,+) \longrightarrow (+,-)\\ (-,-) \longrightarrow (-,-)\end{array}$$

$$T: \qquad (+,+) \longrightarrow (+,+), \qquad \begin{array}{c}(+,-) \longrightarrow (+,-)\\ (-,+) \longrightarrow (-,-)\\ (-,-) \longrightarrow (-,+)\end{array}$$

- Even and odd spin structures transform irreducibly under modular transformations.
- Invariance under global diffeomorphisms $\Longrightarrow$ Need modular invariant expression for the partition function!

Partition function

Recall the bosonic partition function on a torus:

$$Z(\tau) = \operatorname{Tr} e^{2\pi i \tau H}$$

Generalize this to the fermionic field:

$$Z^{++}(\tau) = \eta_{++} \operatorname{Tr} e^{2\pi i \tau H_R} (-1)^F \qquad Z^{+-}(\tau) = \eta_{+-} \operatorname{Tr} e^{2\pi i \tau H_R}$$

$$Z^{-+}(\tau) = \eta_{-+} \operatorname{Tr} e^{2\pi i \tau H_{NS}} (-1)^F \qquad Z^{--}(\tau) = \eta_{--} \operatorname{Tr} e^{2\pi i \tau H_{NS}}$$

On symbols and notations:

- $Z^{\pm\pm}$: contribution of the corresponding spin structure for a left-mover
- $\eta_{\pm\pm}$: phase factor which ensures the modular invariant combination
- $(-1)^F$: For anticommuting variables, the trace ensures antiperiodicity over ξ^2, so insertion of $(-1)^F$ will result in the desired periodicity.
- $H_R = \sum_{m=1}^{\infty} m d^i_{-m} d^i_m + \frac{1}{3}$ and $H_{NS} = \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r - \frac{1}{6}$

Partition function

After some calculations:

$$Z^{++}(\tau) = \eta_{++}\frac{\theta_1^4(\tau)}{\eta^4(\tau)} \qquad Z^{+-}(\tau) = \eta_{+-}\frac{\theta_2^4(\tau)}{\eta^4(\tau)}$$

$$Z^{-+}(\tau) = \eta_{-+}\frac{\theta_4^4(\tau)}{\eta^4(\tau)} \qquad Z^{--}(\tau) = \eta_{--}\frac{\theta_3^4(\tau)}{\eta^4(\tau)}$$

where $\theta_i(\tau)$ are the Jacobi theta functions and η is Dedekind's eta function.

- $\theta_1 \equiv 0$
- Recall the modular transformation propterties

$$\theta_2\left(-{}^1/_\tau\right) = (-i\tau)^{1/2}\theta_4(\tau) \qquad \theta_2(\tau+1) = e^{i\pi/4}\theta_2(\tau)$$

$$\theta_3\left(-{}^1/_\tau\right) = (-i\tau)^{1/2}\theta_3(\tau) \qquad \theta_3(\tau+1) = \theta_4(\tau)$$

$$\theta_4\left(-{}^1/_\tau\right) = (-i\tau)^{1/2}\theta_2(\tau) \qquad \theta_4(\tau+1) = \theta_3(\tau)$$

$$\eta\left(-{}^1/_\tau\right) = (-i\tau)^{1/2}\eta(\tau) \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

Partition function

Determining phases by modular invariance

Determine the phase factors by requiring modular invariance of the partition function separately for left- and right-movers!

$$Z(\tau) = Z_{\text{ferm}}(\tau) Z_{\text{bos}}(\tau) = \left[Z^{++}(\tau) + Z^{+-}(\tau) + Z^{-+}(\tau) + Z^{--}(\tau) \right] Z_{\text{bos}}(\tau)$$

- Recall: $Z_{\text{bos}}(\tau) \propto {}^1/\eta^8(\tau)$
- Do modular transformations! (Phase factors coming from Z_{bos} also included.)

$$Z^{+-}(\tau+1) = Z^{+-}(\tau)$$

$$Z^{-+}(\tau+1) = -\eta_{-+} \frac{\theta_3^4(\tau)}{\eta(\tau)} \stackrel{!}{=} Z^{--}(\tau)$$

$$Z^{--}(\tau+1) = -\eta_{--} \frac{\theta_4^4(\tau)}{\eta(\tau)} \stackrel{!}{=} Z^{-+}(\tau)$$

$$Z^{+-}(-{}^1/\tau) = \eta_{+-} \frac{\theta_4^4(\tau)}{\eta(\tau)} \stackrel{!}{=} Z^{-+}(\tau)$$

$$Z^{-+}(-{}^1/\tau) = \eta_{-+} \frac{\theta_2^4(\tau)}{\eta(\tau)} \stackrel{!}{=} Z^{+-}(\tau)$$

$$Z^{--}(-{}^1/\tau) = Z^{--}(\tau)$$

$$\Longrightarrow \eta_{+-} = \eta_{-+} = -\eta_{--} \stackrel{!}{=} -1$$

Z^{++} transforms irreducibly, η_{++} cannot be determined like this, but further considerations show that $\eta_{++} = \pm 1$

Partition function

Acquiring the GSO projection

The partition function of the worldsheet fermions:

$$Z_{\text{ferm}} = \text{Tr}\, e^{2\pi i \tau H_{NS}} \underbrace{\frac{1}{2}\left(1-(-1)^F\right)}_{\text{GSO projection in the NS-sector}} - \text{Tr}\, e^{2\pi i \tau H_R} \underbrace{\frac{1}{2}\left(1 \pm (-1)^F\right)}_{\text{GSO projection in the R-sector}} =$$

$$= \frac{1}{2\eta^4(\tau)} \left[\underbrace{\theta_3^4(\tau) - \theta_4^4(\tau) - \theta_2^4(\tau)}_{\text{Jacobi identity } = 0} \pm \underbrace{\theta_1^4(\tau)}_{\theta_1 \equiv 0} \right] = 0$$

$$\Longrightarrow \; Z(\tau) = Z_{\text{ferm}}(\tau) Z_{\text{bos}}(\tau) = 0$$

- GSO projection
- Equal contribution from spacetime bosons (NS) and fermions (R): partition function vanishes. A necessary condition for spacetime supersymmetry.

Outline

Supersymmetry

Point particles

Spacetime supersymmetric point particle action

$$S = \frac{1}{2} \int d\tau \, e^{-1} (\dot{x}^\mu - i\bar{\theta}^A \Gamma^\mu \dot{\theta}^A)^2$$

- $\theta^{Aa}(\tau)$ are Grassmann odd variables
- N supersymmetry: label $A = 1, \ldots, N$
- Spinor index $a = 1, \ldots, 2^{D/2}$
- Symmetries of the action (besides global Lorentz, local reparametrization and Weyl invariance):

global supersymmetry:

$$\delta\theta^A = \epsilon^A$$
$$\delta x^\mu = i\bar{\epsilon}^A \Gamma^\mu \theta^A$$
$$\delta e = 0$$

local fermionic symmetry:

$$\theta^A = i\Gamma \cdot p\kappa^A$$
$$\delta x^\mu = i\bar{\theta}^A \Gamma^\mu \delta\theta^A$$
$$\delta e = 4e\bar{\theta}^A \kappa^A$$

local bosonic symmetry:

$$\delta\theta^A = \lambda\dot{\theta}^A$$
$$\delta x^\mu = i\bar{\theta}^A \Gamma^\mu \delta\theta^A$$
$$\delta e = 0$$

Supersymmetry
Strings

Spacetime supersymmetric string action

$$S = S_1 + S_2$$

$$S_1 = -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{h}h^{\alpha\beta}\Pi_\alpha\cdot\Pi_\beta, \qquad \text{where} \qquad \Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A$$

$$S_2 = \frac{1}{\pi}\int d^2\sigma\,\left\{-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\right\}$$

- S_2 is needed for *local fermionic symmetry*, that only works if $N = 0, 1, 2$
- S_2 is *supersymmetric* in these cases:
  1. $D = 3$ and θ is Majorana
  2. $D = 3$ and θ is Majorana or Weyl
  3. $D = 6$ and θ is Weyl
  4. **D = 10 and θ is Majorana–Weyl**

Supersymmetric string theories

Majorana–Weyl spinor *implies* definite chirality for θ^1 and θ^2.

Type I: Open superstring theory, only one chirality is posible due to boundary conditions. This results in $N = 1$.

Type IIA: Closed superstring theory, where θ^1 and θ^2 have opposite chirality. $N = 2$

Type IIB: Closed superstring theory, where θ^1 and θ^2 have the same chirality. $N = 2$

Heterotic: Using only one θ coordinate.

Gauge fixing

- Weyl and reparametrization invariance to fix $h_{\alpha\beta} = \eta_{\alpha\beta}$
- The remaining symmetries to enforce light cone gauge. The degrees of freedom of θ^1 and θ^2:

$$\text{32 complex} \xrightarrow{\text{Majorana}} \text{32 real} \xrightarrow{\text{Weyl}} \text{16 real} \xrightarrow{\text{light cone gauge}} \text{8 real}$$

- Light cone gauge still has rotational invariance for the transverse dimensions.
  $\implies$ Surviving eight components of θ^1 and θ^2 can be viewed as spinor representations of SO(8).
- New symbol for the surviving eight: S^1 and S^2.
- Convention: S^{1a} belongs to $\mathbf{8}_\mathbf{s}$.

Type I	Type IIA	Type IIB
S^{2a} is also $\mathbf{8}_\mathbf{s}$	S^{2a} is $\mathbf{8}_\mathbf{c}$	S^{2a} is also $\mathbf{8}_\mathbf{s}$

$$a, \bar{a} = 1, \ldots, 8$$

Light cone gauge

In the light cone gauge, the equations of motion simplify dramatically. To the point, where they can also be obtained from the action:

Green–Schwarz action in light cone gauge

$$S_{GS}^{l.c.} = -\frac{1}{2\pi}\int d^2\sigma\,(\partial_\alpha X^i \partial^\alpha X^i - i\bar{S}^a \rho^\alpha \partial_\alpha S^a)$$

- S^{1a} and S^{2a} were combined into a two-component Majorana worldsheet spinor S^a. Separately, they are one-component Majorana-Weyl spinors on the worldsheet describing left- or right-movers.
- Compare with NSR action in light cone gauge.

$$S_{NSR}^{l.c.} = -\frac{1}{2\pi}\int d^2\sigma\,\left(\partial_\alpha X^i \partial^\alpha X^i - i\bar{\psi}^i \rho^\alpha \partial_\alpha \psi^i\right)$$

- Both ψ^i and S^a are worldsheet spinors but ψ^i is a $\mathbf{8_v}$, and S^a is a $\mathbf{8_{s/c}}$ representation of SO(8).

Boundary conditions, quantization

Quantization is really similar to the NSR formulation.

$$\{S^{Aa}(\sigma,\tau), S^{Bb}(\sigma',\tau)\} = \pi\delta^{ab}\delta^{AB}\delta(\sigma-\sigma')$$

Boundary conditions can destroy supersymmetry, keep as many as possible!

Open strings: equate at boundaries

$$S^{1a}(0,\tau) = S^{2a}(0,\tau)$$
$$S^{1a}(\pi,\tau) = S^{2a}(\pi,\tau)$$
$$S^{1a} = \frac{1}{\sqrt{2}}\sum S_n^a e^{-in(\tau-\sigma)}$$
$$S^{1a} = \frac{1}{\sqrt{2}}\sum S_n^a e^{-in(\tau+\sigma)}$$

Supersymmetry reduces to $N=1$
type I theory

Closed strings: periodic boundary

$$S^{1a}(\sigma,\tau) = S^{1a}(\sigma+\pi,\tau)$$
$$S^{2a}(\sigma,\tau) = S^{2a}(\sigma+\pi,\tau)$$
$$S^{1a}(\sigma,\tau) = \sum S_n^a e^{-2in(\tau-\sigma)}$$
$$S^{2a}(\sigma,\tau) = \sum \tilde{S}_n^a e^{-2in(\tau+\sigma)}$$

Keeps $N=2$ supersymmetry
type II theory

$$S^a_{-m} = (S^a_m)^\dagger$$

$$\{S^a_m, S^b_n\} = \delta^{ab}\delta_{m+n}$$

Triality

Equivalence of the NSR and GS formulations

Triality: There exists an automorphism of SO(8), that permutes the representations $\mathbf{8_v}, \mathbf{8_s},$ and $\mathbf{8_c}$.

bosonization $\longrightarrow$ reshuffling $\longrightarrow$ refermionization

$$\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_1 = \bar{\psi}^1\rho^\alpha\psi^2 \qquad \sigma_1 = \frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4) \qquad \frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_1 = \bar{S}^1\rho\alpha S^2$$

$$\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_2 = \bar{\psi}^3\rho^\alpha\psi^4 \qquad \sigma_2 = \frac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4) \qquad \frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_2 = \bar{S}^3\rho\alpha S^4$$

$$\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_3 = \bar{\psi}^5\rho^\alpha\psi^6 \qquad \sigma_3 = \frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4) \qquad \frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_3 = \bar{S}^5\rho\alpha S^6$$

$$\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_4 = \bar{\psi}^7\rho^\alpha\psi^8 \qquad \sigma_4 = \frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4) \qquad \frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_4 = \bar{S}^7\rho\alpha S^8$$

The Neveu–Schwarz–Ramond formulation (with GSO projection) is equivalent to the Green–Schwarz one.

Ground state

- The mass^2 operator:

$$\alpha' m^2 = \underbrace{\sum_{n>0} \alpha^i_{-n}\alpha^i_n}_{N^{(\alpha)}} + \underbrace{\sum_{m>0} m S^a_{-m} S^a_m}_{N^{(S)}}$$

No normal ordering constant $\implies$ ground state is massless.

- Ground state degeneracy: S_0 maps ground state to ground state and it obeys the Clifford algebra:

$$\{S^a_0, S^b_0\} = \delta^{ab}$$

Triality, similar construction to R-sector in NSR, just $\mathbf{8_v} \leftrightarrow \mathbf{8_{s/c}}$

$$S^a_0 \propto \gamma^a = \begin{pmatrix} 0 & \gamma^a_{i\,\bar{a}} \\ \gamma^a_{\bar{a}i} & 0 \end{pmatrix} \qquad \text{or} \qquad S^{\bar{a}}_0 \propto \gamma^{\bar{a}} = \begin{pmatrix} 0 & \gamma^{\bar{a}}_{i\,a} \\ \gamma^{\bar{a}}_{ai} & 0 \end{pmatrix}$$

The ground state is now a $\mathbf{8_v} \oplus \mathbf{8_{c/s}}$ multiplet:

$$|\phi_0\rangle_{\mathbf{8_v}\oplus\mathbf{8_c}} = |i\rangle\, \zeta^i(k) + |\bar{a}\rangle\, \lambda^{\bar{a}}_c(k) \qquad \text{or} \qquad |\phi_0\rangle_{\mathbf{8_v}\oplus\mathbf{8_s}} = |i\rangle\, \zeta^i(k) + |a\rangle\, \lambda^a_s(k)$$

Open superstring spectrum

Ground state is $\mathbf{8_v} \oplus \mathbf{8_c}$

$\alpha' m^2$	states and their SO(8) representation contents	little group	representation contents with respect to the little group
0	$\|i\rangle$ $\mathbf{8_v}$ $\|\bar{a}\rangle$ $\mathbf{8_c}$	SO(8)	$\mathbf{8_v}$ (boson) $\mathbf{8_c}$ (fermion)
$+1$	$\alpha^j_{-1}\,\|i\rangle$ $\quad S^b_{-1}\,\|\bar{a}\rangle$ $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$ $\quad \mathbf{8_v} \oplus \mathbf{56_v}$ $\alpha^i_{-1}\,\|\bar{a}\rangle$ $\quad S^b_{-1}\,\|i\rangle$ $\mathbf{8_s} \oplus \mathbf{56_s}$ $\quad \mathbf{8_c} \oplus \mathbf{56_c}$	SO(9)	$\mathbf{84} \oplus \mathbf{44}$ (bosons) $\mathbf{128}$ (fermions)

Compare with the result of the NSR formulation with GSO projection!

$\alpha' m^2$	states and their SO(8) representation contents	little group	representation contents with respect to the little group
NS-sector (bosons)			
0	$b^i_{-1/2}\,\|0\rangle$ $\mathbf{8_v}$	SO(8)	$\mathbf{8_v}$
$+1$	$b^i_{-1/2} b^j_{-1/2} b^k_{-1/2}\,\|0\rangle_{i}$ $\mathbf{56_v}$ $\alpha^i_{-1} b^j_{-1/2}\,\|0\rangle$ $\quad b^i_{-3/2}\,\|0\rangle$ $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}$ $\quad \mathbf{8_v}$	SO(9)	$\mathbf{84} \oplus \mathbf{44}$
R-sector (fermions)			
0	$\|\bar{a}\rangle$ $\mathbf{8_c}$	SO(8)	$\mathbf{8_c}$
$+1$	$\alpha^i_{-1}\,\|\bar{a}\rangle$ $\quad d^i_{-1}\,\|a\rangle$ $\mathbf{8_s} \oplus \mathbf{56_s}$ $\quad \mathbf{8_c} \oplus \mathbf{56_c}$	SO(9)	$\mathbf{128}$

Closed superstring spectrum

Closed superstring states are tensor products of left- and right-movers.
For the massless level:

Type IIA: Opposite chirality for left- and right-movers.

$$(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) = (\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \oplus \mathbf{8_v} \oplus \mathbf{56_v})_{\text{bosonic}} \oplus (\mathbf{8_s} \oplus \mathbf{8_c} \oplus \mathbf{56_s} \oplus \mathbf{56_c})_{\text{fermionic}}$$

Type IIB: Same chirality for the left- and right-movers.

$$(\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) = (\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c})_{\text{bosonic}} \oplus (\mathbf{8_s} \oplus \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{56_s})_{\text{fermionic}}$$

Agrees with the NSR formulation!

Thank you!

THANK YOU FOR YOUR ATTENTION!