ETH
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The Ising model in Conformal Field Theory

March11th 2013 | Theoretisches Proseminar

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Outline

- Motivation**:** → CFT "in action": physical application!
- **Part I**: Overview of statistical phyisics and the 2D Ising model
- **Part II:** From the 2D Ising model to the free fermion.
  - Step 1**:** classical to quantum correspondence.
  - Step 2: Jordan-Wigner transformation.
  - Step 3: exact solution and continuum limit.
- **Part III:** conformal field theory for the free fermion.

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PART I: basics of statistical mechanics

- Complexity → microstate/macrostate formulation.
- Boltzmann Distribution and partition function.

$$\mathbb{P}_n = \frac{1}{Z} e^{-\beta E_n} \qquad Z = \sum_n e^{-\beta E_n} = \sum_n \langle \psi_n | \, e^{-\beta \hat{H}} \, | \psi_n \rangle = \mathrm{Tr}\, \rho$$

- Z as thermodynamic generating function:

$$U = \frac{1}{Z} \sum_i E_i e^{-\beta E_i} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}$$

$$F = U - TS = -T \log Z$$

$$M = - \left. \frac{\partial F}{\partial h} \right|_T$$

PART I: Phase transitions and criticality

- Phase transitions = sudden change in macroscopic properties of the system as a control parameter is varied (e.g. T, p).

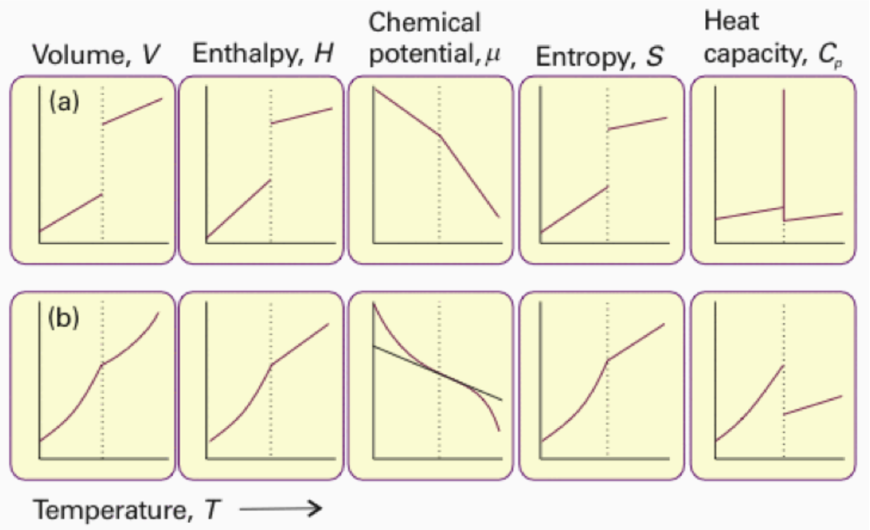
  ➔ condensation, evaporation, sublimation, …

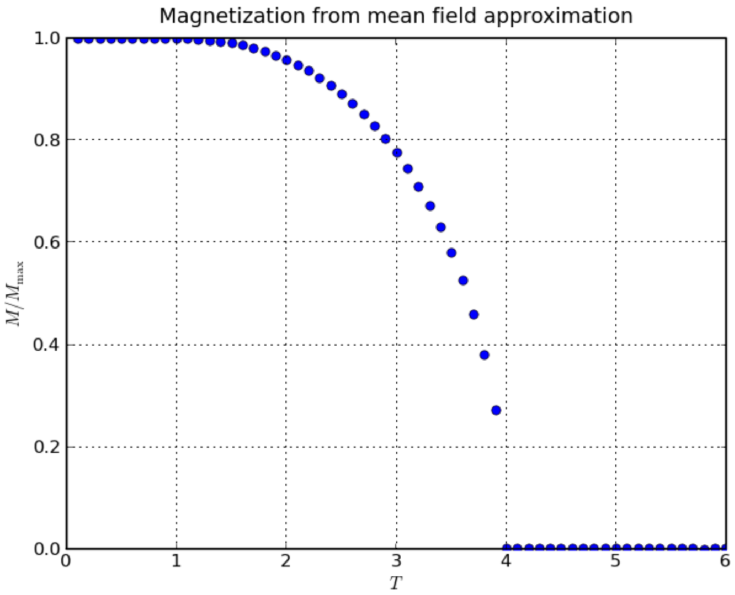
  ➔ superconductivity

  ➔ ferromagnetic/paramagnetic transition at Curie temperature (Ising!)

- Distinction between:
  - First order phase transition → latent heat, finite jump in U
  - <u>Second order phase transition</u> → derivatives of macroscopic quantities discontinuous, e.g. Ising: χ
- Order parameter: distinguishes different phases, e.g. Ising: M

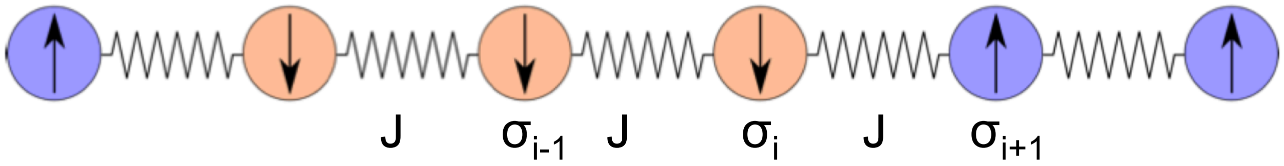
PART I: Phase transitions and criticality



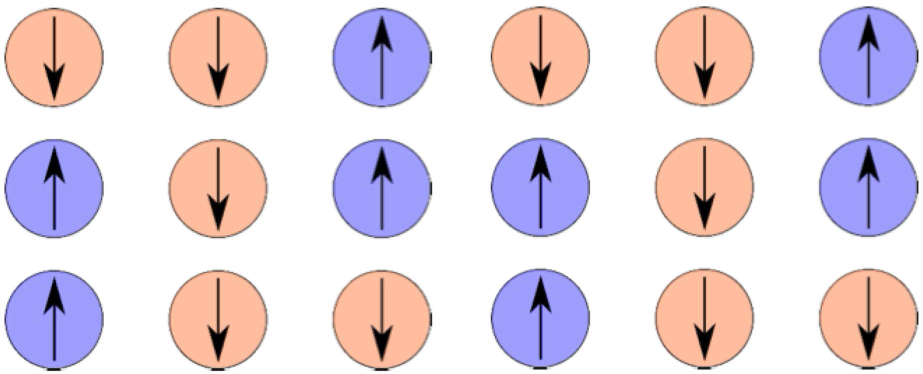


PART I: the classical Ising model

- Configuration energy: $E[\sigma] = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$ $\qquad \sigma_i = \pm 1$
- 1D:



- 2D:



- Simple model for ferromagnetism.
- 2D model solved exactly by Onsager for h=0 (1944). Case h ≠ 0 ?

PART I: the classical Ising model

- Configuration energy: $E[\sigma] = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$ $\qquad \sigma_i = \pm 1$

- Magnetization: $M = \langle \sigma_j \rangle = \frac{1}{Z} \sum_{[\sigma]} \left\{ \frac{1}{N} \sum_i \sigma_i \right\} e^{-\beta E[\sigma]}$

- Susceptibility: $\chi = \left. \frac{\partial M}{\partial h} \right|_{h=0}$ ➔ $\boxed{\chi \propto \mathrm{Var}(\sigma_{\mathrm{tot}})}$ $\qquad \sigma_{\mathrm{tot}} = \sum_i \sigma_i$

- Variance → pair correlation function: $\Gamma_c(i-j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$

- Susceptibility as a measure of the statistical fluctuations of the dipole moment: $\chi = \beta \sum_i \Gamma_c(i)$

PART I: generalizations of the Ising model

- Rewrite spin-spin interaction: $\sigma_i \sigma_j = 2\delta_{\sigma_i,\sigma_j} - 1$
- q-Potts models: spins take values 0,...,q-1.

$$E[\sigma] = -2J \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} - h \sum_i \sigma_i$$

- Replace spins with unit vectors → Heisenberg model

$$E[\mathbf{n}] = J \sum_{\langle ij \rangle} \mathbf{n}_i \cdot \mathbf{n}_j - \sum_i \mathbf{h} \cdot \mathbf{n}_i$$

- Continuum limit of the lattice → φ^4-model, case u=0 exactly solvable

$$E[\mathbf{n}] = \int \mathrm{d}^m x \left\{ \frac{1}{2} \partial_k \mathbf{n} \cdot \partial_k \mathbf{n} - \frac{1}{2} \mu^2 \mathbf{n}^2 + \frac{1}{4} u (\mathbf{n}^2)^2 \right\}$$

PART I: Phase transitions in the ising model

- Kramers-Wannier duality relation:

$$\sinh\left(\frac{2J}{T_c}\right) = 1 \quad \rightarrow \quad T_c = \frac{2J}{\log(1+\sqrt{2})}$$

- Magnetization M:
  - M=0 → symmetric phase (spins are not aligned)
  - M≠0 → ordered phase (spins are aligned)
- Discrete Z_2 symmetry breaking: $E[\sigma] = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i$
  - Reversal of spins: $\sigma_i \rightarrow -\sigma_i$
  - <Q> ≠ 0 for quantity Q not invariant under symmetry.
  - M=$\langle\sigma_i\rangle$ simplest of these Q → *order parameter*

PART I: Peierls droplets in the Ising model

- http://www.pha.jhu.edu/~javalab/ising/ising.html
- http://physics.ucsc.edu/~peter/ising/ising.html

PART I: critical exponents

- Behavior of physical quantities as $T \rightarrow T_c$?

$$M \sim (T_c - T)^{\frac{1}{8}} \qquad \chi = \frac{\partial M}{\partial h} \sim (T - T_c)^{-\frac{7}{4}}$$

- Correlation length ξ:

$$\Gamma(i-j) \sim e^{-\frac{|i-j|}{\xi(T)}}, \qquad |i-j| >> 1$$

$$\xi(T) \sim \frac{1}{|T - T_c|} \rightarrow \infty, \qquad \text{as } T \rightarrow T_c$$

- Correlation length can exceed system's dimension L

→ algebraic decay $\Gamma(n) \sim \frac{1}{|n|^{d-2+\eta}}$

PART I: critical exponents

Table 3.1. Definitions of the most common critical exponents and their exact value within the two-dimensional Ising model. Here d is the dimension of space.

Exponent	Definition	Ising Value
α	$C \propto (T - T_c)^{-\alpha}$	0
β	$M \propto (T_c - T)^{\beta}$	1/8
γ	$\chi \propto (T - T_c)^{-\gamma}$	7/4
δ	$M \propto h^{1/\delta}$	15
ν	$\xi \propto (T - T_c)^{-\nu}$	1
η	$\Gamma(n) \propto \|n\|^{2-d-\eta}$	1/4

PART I: universality

- **Universality**: a system shows universality when ist order parameter stops depending on local (microscopic) details once the system is close enough to criticality.
  - Ising: block spin renormalization (blackboard)

    → magnetization does not depend on the lattice geometry
- Universality between different phenomena:
  - They share the same set of critical exponents.
  - *Universality classes*: ferromagnetic transition (Ising), percolation of coffee, critical opalescence of liquid, …
- Formal theoretical explanation: renormalization group theory.

PART I: Widom's scaling

- **Scaling hypothesis (Widom):** the free energy density (or per site) near the critical point is a homogeneous function of its parameters h (external field) and t (reduced temperature $t=T/T_c - 1$)

$$f(\lambda^a t, \lambda^b h) = \lambda f(t,h) \quad \rightarrow \quad f(t,h) = t^{\frac{1}{a}} g(y), \qquad y = h t^{-\frac{b}{a}}$$

- → **relate critical exponents to each other!**

$$M = -\frac{\partial f}{\partial h}\bigg|_{h=0} = t^{\frac{1-b}{a}} g'(0)$$

$$\chi = \frac{\partial^2 f}{\partial h^2}\bigg|_{h=0} = t^{\frac{1-2b}{a}} g''(0)$$

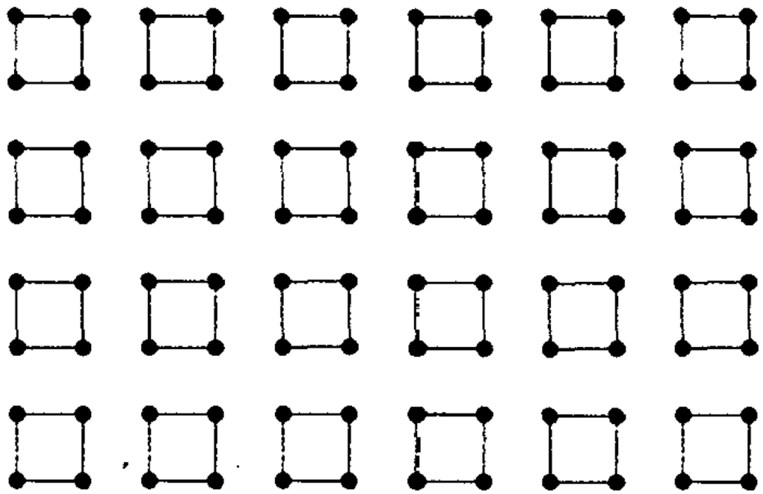
$$\alpha = 2 - \frac{1}{a}$$

$$\beta = \frac{1-b}{a}$$

$$\gamma = -\frac{1-2b}{a}$$

$$\delta = \frac{b}{1-b}$$

PART I: block spin renormalization



block spins of side r (here r=2)

→ r^d spins grouped in one (here 4)

PART I: Block spin renormalization

- Aim → justify Widom's law: $f(\lambda^a t, \lambda^b h) = \lambda f(t, h)$
- Group spin: $\Sigma_I = \frac{1}{R} \sum_{i \in I} \sigma_i$
- New Hamiltonian: $H' = -J' \sum_{\langle IJ \rangle} \Sigma_I \Sigma_J - h' \sum_I \Sigma_I$
- Total free energy should not be affected by our grouping procedure:

$$f(t, h) = r^{-d} f(r^{\frac{1}{\nu}} t, Rh)$$

PART I: Block spin renormalization

$$\Gamma'(n) = \langle \Sigma_I \Sigma_J \rangle - \langle \Sigma_I \rangle \langle \Sigma_J \rangle =$$

$$= R^{-2} \sum_{i \in I} \sum_{j \in J} \{ \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \} =$$

$$= R^{-2} \, r^{2d} \, \Gamma(rn) = \frac{R^{-2} r^{2d}}{|rn|^{d-2+\eta}}$$

$$= \frac{R^{-2} r^{d+2-\eta}}{|n|^{d-2+\eta}}$$

$$\boxed{f(r^{\frac{1}{\nu}} t, r^{\frac{d+2-\eta}{2}} h) = r^d f(t, h)}$$

➔ $a = \dfrac{1}{\nu d}, \qquad b = \dfrac{d+2-\eta}{2d}$

PART I: Block spin renormalization

Rushbrooke's law	$\alpha + 2\beta + \gamma$
Widom's law	$\gamma = \beta(\delta - 1)$
Fisher's law	$\gamma = \nu(2 - \eta)$
Josephson's law	$\nu d = 2 - \alpha$

Table 2: Summary of the scaling laws [4].

- Critical exponents can be expressed through ν and η

  → relate all physical quantities at criticality to correlation functions!

  → **Quantum field theory → Conformal field theory**

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PART II - Step 1: 1D statistical quantum Ising model

- Canonical quantization:
  - Observables → operators
  - Results of measurements → eigenvalues
  - Phase coordinates → (eigen)states
  - **Ising: configuration energy → Hamiltonian**

$$\mathcal{H} = H_0 + H_1 = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

- "Quantum Ising model in a transverse field".
- Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

PART II - Step 1: Pauli spin operator algebra

- Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Involution:

$$(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = \mathbb{1}$$

- Commutation relations:

$$[\sigma^a, \sigma^b] = 2i\,\epsilon_{abc}\,\sigma^c$$

$$\{\sigma^a, \sigma^b\} = 2\delta_{ab}\,\mathbb{1}$$

PART II - Step 1: Pauli spin operator algebra

- Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Eigenstates : $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- Completeness relations: $\sum_{S^a=\pm1} |S^a\rangle \langle S^a| = 1$

- Action of σ^x on σ^z-eigenstates: $\sigma^x |+\rangle = |-\rangle, \qquad \sigma^x |-\rangle = |+\rangle$

PART II - Step 1: time slicing

- Partition function for the quantum 1D Ising model:

$$Z = \mathrm{Tr} e^{-\beta\mathcal{H}} = \mathrm{Tr}\left[e^{-\Delta\tau\mathcal{H}} e^{-\Delta\tau\mathcal{H}} \cdots e^{-\Delta\tau\mathcal{H}}\right]$$

→ "imaginary time evolution"

- Insert completeness relations: $\prod_{i=1}^{N}\left[\sum_{S_i^z=\pm1} |S_i^z\rangle\langle S_i^z|\right] \equiv \sum_{\{S_i^z\}} |S^z\rangle\langle S^z| = 1$

- New labeling for time interval l: $\sum_{\{S_{i,l}^z\}} |S_l^z\rangle\langle S_l^z| = 1$

$$\rightarrow \quad Z = \sum_{\{S_{i,l}=\pm1\}} \langle S_1^z| e^{-\Delta\tau\mathcal{H}} |S_L^z\rangle \langle S_L^z| e^{-\Delta\tau\mathcal{H}} |S_{L-1}^z\rangle \cdots \langle S_2^z| e^{-\Delta\tau\mathcal{H}} |S_1^z\rangle$$

PART II - Step 1: Suzuki-Trotter formula

- Look at one matrix element: $\langle S^z_{l+1} | e^{-\Delta\tau\mathcal{H}} | S^z_l \rangle$
- Problem: H_0 and H_1 do not commute ☹.

  → Lie –Trotter formula: $$e^{A+B} = \lim_{L\to\infty} \left(e^{A/L} e^{B/L} \right)^L$$
- "Finite version" of Lie-Trotter:

  → Suzuki-Trotter approximation:

$$e^{-\Delta\tau H_0 - \Delta\tau H_1} = e^{-\Delta\tau H_0} e^{-\Delta\tau H_1} + \mathcal{O}((\Delta\tau)^2 [H_0, H_1])$$

- Justification:

$$(\Delta\tau)^2 Jh \ll 1 \quad \rightarrow \quad L \gg \beta\sqrt{Jh}$$

PART II - Step 1: Suzuki-Trotter formula cont'd

- Apply Suzuki-Trotter to $\left\langle S_{l+1}^z \right| e^{-\Delta\tau\mathcal{H}} \left| S_l^z \right\rangle$
- Drop the $\Delta\tau^2$-proportional term:

$$\left\langle S_{l+1}^z \right| e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} \left| S_l^z \right\rangle = \left\langle S_{l+1}^z \right| e^{-\Delta\tau H_1} e^{-\Delta\tau J \sum_{i=1}^N \sigma_{i,l}^z \sigma_{i+1,l}^z} \left| S_l^z \right\rangle$$

$$= e^{-\Delta\tau J \sum_{i=1}^N S_{i,l}^z S_{i+1,l}^z} \underbrace{\left\langle S_{l+1}^z \right| e^{-\Delta\tau h \sum_{i=1}^N \sigma_i^x} \left| S_l^z \right\rangle}_{\text{Still to evaluate!}}$$

Still to evaluate!

PART II - Step 1: Pauli matrix exponential

- Evaluate: $\left\langle S^z_{l+1} \right| e^{-\Delta\tau h \sum_{i=1}^{N} \sigma_i^x} \left| S^z_l \right\rangle$
- Use involutory property: $(\sigma_i^x)^2 = \mathbb{1}$

→ $$e^{\Delta\tau h \sigma_i^x} = \mathbb{1}\cosh(\Delta\tau h) + \sigma_i^x \sinh(\Delta\tau h)$$

- Bring matrix element in the $e^{-\beta H}$ form of classical Z:

$$\left\langle \tilde{S}^z \right| e^{\Delta\tau h \sigma_i^x} \left| S^z \right\rangle \equiv \Lambda e^{\gamma \tilde{S}^z S^z}$$

- Determine Λ and γ by using eigenstates:

$$\langle + | \, e^{\Delta\tau h \sigma_i^x} \, | + \rangle = \cosh(\Delta\tau h) = \Lambda e^{\gamma}$$

$$\langle - | \, e^{\Delta\tau h \sigma_i^x} \, | + \rangle = \sinh(\Delta\tau h) = \Lambda e^{-\gamma}$$

→ $$\gamma = -\frac{1}{2}\log(\tanh(\Delta\tau h))$$

$$\Lambda^2 = \sinh(\Delta\tau h)\cosh(\Delta\tau h)$$

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PART II - Step 1: anisotropic 2D classical Ising model

- Put everything back together:

$$\left\langle S^z_{l+1}\right| e^{-\Delta\tau H_1} e^{-\Delta\tau H_0} \left|S^z_l\right\rangle = \Lambda^N e^{\Delta\tau J \sum_{i=1}^{N} S^z_{i,l} S^z_{i+1,l} + \gamma \sum_{i=1}^{n} S^z_{i,l} S^z_{i,l+1}}$$

$$\rightarrow \boxed{Z = \Lambda^{NL} \sum_{\{S^z_{i,l}=\pm 1\}} e^{\Delta\tau J \sum_{i=1}^{N} \sum_{l=1}^{L} S^z_{i,l} S^z_{i+1,l} + \gamma \sum_{i=1}^{N} \sum_{l=1}^{L} S^z_{i,l} S^z_{i,l+1}}}$$

- See any anology with the following?

$$Z_{cl} = \Lambda^{NL} \sum_{\{\sigma^z_{i,l}=\pm 1\}} e^{\tilde{\beta} J_x \sum_{i=1}^{N_x} \sum_{l=1}^{N_y} \sigma_{i,l} \sigma_{i+1,l} + \tilde{\beta} J_y \sum_{i=1}^{N_x} \sum_{l=1}^{N_y} \sigma_{i,l} \sigma_{i,l+1}}$$

- Identifications:

$$\sigma_{i,l} = S^z_{i,l} \qquad N_x = N \qquad \tilde{\beta} J_x = \Delta\tau J$$

$$N_y = L \qquad \tilde{\beta} J_y = \gamma$$

PART II - Step 1: remarks

- 1D quantum → 2D classical.
- 2D classical ?→? 1D quantum.
  - Yes! Trick: Write Z as a trace over matrix product.
  - transfer matrices → operators arise naturally from canonical quantization
  - spin transfer → imaginary time step
  - (details in report)
- Generalization: d quantum ⇔ (d + 1) classical
  - Quantum transverse field h induces coupling between different times

    → additional dimension!

    → classical (d+1)-dimensional model is field-free!

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PART II – Step 2: spins in terms of fermions

- So far: 2D classical Ising model ⇔ 1D quantum Ising model
- Now: 1D quanrum Ising model ⇔ free fermion
- Why fermions? Simple mapping between models with spin-½ degrees of freedom per site and spinless fermion hopping between sites with single orbitals

  → spin-up ⇔ empty orbital, → spin-down ⇔ occupied orbital
- Creation/annihilation operators for fermions: $c_i, \quad c_i^\dagger, \quad n_i \equiv c_i^\dagger c_i$

→ Operator relations:

$$\hat{\sigma}_i^z = 1 - 2c_i^\dagger c_i$$

$$\hat{\sigma}_i^+ = c_i \qquad\qquad \hat{\sigma}_i^+ \equiv \frac{1}{2}(\hat{\sigma_i^x} + i\sigma_i^y), \quad \hat{\sigma}_i^- \equiv \frac{1}{2}(\hat{\sigma_i^x} - i\sigma_i^y)$$

$$\hat{\sigma}_i^- = c_i^\dagger$$

PART II – Step 2: the Jordan-Wigner transformation

- Operator relations work for one site: $\{c_i^\dagger, c_i\} = \{\hat{\sigma}_i^-, \hat{\sigma}_i^+\} = 1$
- Naive generalization to the chain → failure!

  → Why? Spin operators commute, fermionic operators anticommute!
- Jordan-Wigner transformation: ⚠ highly non-local!

$$\hat{\sigma}_i^+ = \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i \qquad \Leftrightarrow \qquad c_i = \left(\prod_{j<i}\hat{\sigma}_j^z\right)\hat{\sigma}_i^+$$

$$\hat{\sigma}_i^- = \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i^\dagger \qquad \text{→ use inductively involution of } \sigma^z \text{ →} \qquad c_i^\dagger = \left(\prod_{j<i}\hat{\sigma}_j^z\right)\hat{\sigma}_i^-$$

- Commutators/anticommutators are preserved:

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$$

$$[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \delta_{ij}\hat{\sigma}_i^z, \qquad [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2\delta_{ij}\hat{\sigma}_i^\pm$$

PART II – Step 2: Jordan-Wigner for the Ising chain

- Trick → rotate coordinates: $\hat{\sigma}_i^z \to \hat{\sigma}_i^x, \qquad \hat{\sigma}_i^x \to -\hat{\sigma}_i^z$
- Operator relations:

$$\hat{\sigma}_i^x = 1 - 2c_i^\dagger c_i$$
$$\hat{\sigma}_i^z = -\prod_{j<i}(1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger)$$

- Fermionic Hamiltonian:

$$H_I = -\sum_i \left[J \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i \right) - 2h c_i^\dagger c_i + h \right]$$

  - Quadratic? ✓ → diagonalizable (Fourier)
  - Fermionic number conserved? ✗

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PART II – Step 3: Bogoliubov transformation and exact solution

- Discrete Fourier transform: $c_k = \frac{1}{\sqrt{N}} \sum_j c_j e^{ikx}$

➔ $$H_I = \sum_k (2[h - J\cos(ka)]c_k^\dagger c_k + iJ\sin(ka)[c_{-k}^\dagger c_k^\dagger + c_{-k}c_k] - h)$$

- Unitary transformation to a set of operators whose fermionic number is conserved (Bogoliubov transformation):

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger$$

$$u_k^2 + v_k^2 = 1$$
$$v_{-k} = -v_k$$
$$u_{-k} = u_k$$

- Final Hamiltonian: $H_I = \sum_k \epsilon_k(\gamma_k^\dagger \gamma_k - \frac{1}{2})$

- Excitation energy: $\epsilon_k = 2(J^2 + h^2 - 2hJ\cos k)^{\frac{1}{2}}$

PART II – Step 3: continuum limit

- Analyze the behavior of the energy:
  - Dimensionless parameter g: Jg=h →
  - Excitation energy ≥ 0, if h=J → ε_k=0

$$\epsilon_k = 2(J^2 + h^2 - 2hJ\cos k)^{\frac{1}{2}}$$

$$\epsilon_k = 2J\sqrt{(1 + g^2 - 2g\cos k)}$$

- Energy gap (minimum of the excitation energy):

$$\epsilon_{min} = 2J\sqrt{(1 + g^2 - 2g\cos 0)} = 2J\sqrt{((1-g)^2)} = 2J|1-g|$$

  - Vanishes for g=1 ➔ boundary between symmetric and ordered phase!
  - Long wavelength excitation possible with arbitrary low energies → dominate the low-temperature properties.
  - Idea: take continuum limit a → 0 and obtain a continuum quantum field theory in terms of fermions.

PART II – Step 3: continuum limit

- Continuum Fermi fields: $\Psi(x_i) = \frac{1}{\sqrt{a}} c_i$
- Continuum version of anticommutation: $\{\Psi(x), \Psi^\dagger(x')\} = \delta(x - x')$
- Hamiltonian of the free field:

$$H_F = E_0 + \int \mathrm{d}x \left[\frac{c}{2} \left(\Psi^\dagger \frac{\partial \Psi^\dagger}{\partial x} - \Psi \frac{\partial \Psi}{\partial x} \right) + \Delta \Psi^\dagger \Psi \right] + \mathcal{O}(a)$$

  - Couplings: $\Delta = 2(J - h), \qquad c = 2Ja$

- Path integral formulation:

$$\mathcal{Z} = \mathrm{Tr} e^{-\frac{H_F}{T}} = \int \mathrm{D}\Psi \mathrm{D}\Psi^\dagger e^{-\int_0^{1/T} \mathrm{d}\tau \mathrm{d}x \mathcal{L}_I}$$

$$\mathcal{L}_I = \Psi^\dagger \frac{\partial \Psi}{\partial \tau} + \frac{c}{2} \left(\Psi^\dagger \frac{\partial \Psi^\dagger}{\partial x} - \Psi \frac{\partial \Psi}{\partial x} \right) + \Delta \Psi^\dagger \Psi$$

$$\epsilon_k = (\Delta^2 + c^2 k^2)^{\frac{1}{2}}$$

→ Conformal invariance for Δ=0: CFT ⇔ criticality

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PART III: conformal invariance

- Recall the correlation function for 2D Ising model at criticality:

$$\Gamma_c(r) \propto \frac{1}{r^{d-2+\eta}} = \frac{1}{r^{\eta}}$$

- Critical exponent η=¼ → want to match this with the help of CFT!
- Last time: conformal group of infinitesimal transformations leaves metric invariant:

$$g'_{\mu\nu}(\mathbf{x}') = \Lambda(x) g_{\mu\nu}(x)$$

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu(\mathbf{x})$$

$$\text{translation}: \quad x'^\mu = x^\mu + a^\mu$$

$$\text{dilation}: \quad x'^\mu = \alpha x^\mu$$

$$\text{rigid rotation}: \quad x'^\mu = M^\mu_\nu x^\nu$$

$$\text{special conformal transformation}: \quad x'^\mu = \frac{x^\mu + b^\mu \mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2}$$

PART III: conformal invariance on correlation functions

- Quasi-primary fields: $\phi(\mathbf{x}) \to \phi'(\mathbf{x}') = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta/d} \phi(\mathbf{x})$
- Two-point correlation function:

$$\langle \phi_1(\mathbf{x_1})\phi_2(\mathbf{x_2})\rangle = \frac{1}{Z}\int [\mathrm{d}\Phi]\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)e^{-\mathcal{S}[\Phi]}$$

- Invariance of action & measure:

➔ $$\langle \phi_1(\mathbf{x_1})\phi_2(\mathbf{x_2})\rangle = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta_1/d}_{x=x_1} \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta_2/d}_{x=x_2} \langle \phi_1'(\mathbf{x_1})\phi_2'(\mathbf{x_2})\rangle$$

- Invariance under scaling, rigid rotation, etc:

➔ $$\langle \phi_1(\mathbf{x_1})\phi_2(\mathbf{x_2})\rangle = \begin{cases} \frac{C_{12}}{(|\mathbf{x}_1-\mathbf{x}_2|)^{2\Delta_1}}, & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2 \end{cases}$$

- 2D: $$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \qquad h = \tfrac{1}{2}(\Delta + s)$$

PART III: correlation functions

- Primary field for the spin with 2-point function:

$$\langle \sigma(r)\sigma(0)\rangle = \frac{1}{r^{2(h_\sigma+\bar{h}_\sigma)}}$$

- Comparison with classical correlation function: $\eta = 2(h_\sigma + \bar{h}_\sigma)$
- This can be matched for $h_\sigma = \bar{h}_\sigma = \frac{1}{16}$

PART III: operator product expansions (OPE's)

- OPE: way of expanding correlation functions.
- Noether's theorem → conserved current $j^\mu = \eta^{\mu\nu}\mathcal{L}\omega_\nu - \omega_\nu \partial^\nu \phi \frac{\mathcal{L}}{\partial(\partial_\mu \phi)}$
- Energy momentum tensor T: $j^\mu = T^{\mu\nu}\omega_\nu$
- OPE for a primary field φ and T

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w})$$

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w}$$

- Central charge c is different for different models:

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

PART III: free boson VS free fermion

	Free boson	Free fermion
action	$S = \frac{1}{2} g \int \mathrm{d}z\mathrm{d}\bar{z} \left\{ \partial_\mu \phi(z,\bar{z}) \partial^\mu \phi(z,\bar{z}) \right\}$	$S = g \int \mathrm{d}^2 x \bar{\psi} \partial_z \bar{\psi} + \psi \partial_{\bar{z}} \psi$
2-point func.	$\langle \phi(x), \phi(y) \rangle = -\frac{1}{4\pi g} \log(\mathbf{x}-\mathbf{y})^2$	$\langle \psi(z,\bar{z}) \psi(w,\bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{z-w}$
E-M tensor	$T(z) = -2\pi g : \partial\phi\partial\phi :$	$T(z) = -\pi g : \psi(z)\partial\psi(z) :$
OPE's	$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w})$ $T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)}$ $T(z)T(w) \sim \frac{\frac{1}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w}$	$T(z)\psi(w) = \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}$ $T(z)T(w) = \frac{\frac{1}{4}}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$
Conformal dimension h	h=1	h=1/2
Central charge c	c=1	c=1/2

PART III: twist fields

- Aim: determine conformal dimension of the primary fields for σ:
- Laurent expansion: $i\psi(z) = \sum_n \psi_n z^{-n-h}$ $\psi_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n-h} \psi(z)$
- Anticommutation relations of the modes: $\{\psi_n, \psi_m\} = \delta_{n,-m}$
- Modes act as fermioni creation/annihilation operators:

$$\psi_n |0\rangle = 0, \qquad \psi_{-n_1} \ldots \psi_{-n_k} |0\rangle = |n_1, \ldots, n_k\rangle$$

- Radial quantization → boundary conditions:

$$\text{periodic BC}: \quad \psi(e^{2\pi i} z) = \psi(z) \quad \Rightarrow \quad n \in \mathbb{Z} + \frac{1}{2}$$

$$\text{antiperiodic BC}: \quad \psi(e^{2\pi i} z) = -\psi(z) \quad \Rightarrow \quad n \in \mathbb{Z}$$

- Representation of Virasoro algebra through ψ_0, with anticommutators:

$$\{\psi_0, \bar{\psi}_0\}, \qquad \{\psi_0, \psi_0\} = \{\bar{\psi}_0, \bar{\psi}_0\} = 1$$

- Smallest irreducible rep (operator-state correspondence): $\left|\frac{1}{16}\right\rangle_{\pm}$

PART III: twist fields cont'd

- Action of $\left|\frac{1}{16}\right\rangle_{\pm}$ can be represented by Pauli matrices (same algebra):

$$\bar{\psi}_0 = \frac{1}{\sqrt{2}}\sigma^z, \qquad \psi_0 = \frac{1}{\sqrt{2}}\sigma^x$$

$$\bar{\psi}_0 \left|1/16\right\rangle_{\pm} = \frac{1}{\sqrt{2}}\left|1/16\right\rangle_{\pm}, \qquad \psi_0 \left|1/16\right\rangle_{\pm} = \pm\frac{1}{\sqrt{2}}\left|1/16\right\rangle_{\mp}$$

- The fields associated with $\left|\frac{1}{16}\right\rangle_{\pm}$ are called <u>twist fields:</u>

$$\left|\frac{1}{16}\right\rangle_{+} = \sigma(0)\left|0\right\rangle$$

$$\left|\frac{1}{16}\right\rangle_{-} = \mu(0)\left|0\right\rangle$$

- Determine conformal weight of σ → look at OPE of e-m tensor:

$$\frac{1}{2}\left\langle\sigma(z)\partial_w\sigma(w)\right\rangle_A = \frac{1}{2}\partial_w\left\langle\sigma(z)\sigma(w)\right\rangle_A = -\frac{1}{2(z-w)^2} + \frac{1}{16w^{\frac{3}{2}}z^{\frac{1}{2}}}$$

- On the other hand: $T(z)\sigma(w) = \sum_{n\geq 0}(z-w)^{n-2}L_n\sigma(w)$

→ $$\langle T\rangle_A = \langle 1/16|_+ T(z)\left|1/16\right\rangle_+ = \langle 1/16|_+ \frac{1}{z^2}L_0\left|1/16\right\rangle_+ = \frac{h_\sigma}{z^2}$$

$$\rightarrow \quad h_\sigma = \bar{h}_\sigma = \frac{1}{16} \quad \leftarrow$$

Take home messages:

- Simple model for ferro/paramagnetic phase transition → Ising model
- Critical exponents and idea of universality.
- Correspondence between classical and quantum systems.
- Correspondence between statistical mechanics and QFT.
- Use of symmetries (scaling) in CFT to obtain physically measurable quantities.

➔ CFT as mathematical tools which can be applied to systems at criticality!

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Thank you for your attention!