Modular invariance and orbifolds

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Abstract

One of the simplest Riemann surfaces with nonzero genus is the torus. Modular transformations do not change the complex structure on the torus, and therefore it is possible to study the constraints following from modular invariance on conformal field theories defined on the torus. After a general introduction to the modular group, we discuss different models of conformal field theories on the torus: The partition function of a free boson on a torus is calculated using ζ regularization. Other models, such as the free fermion and the compactified boson, are investigated. Furthermore, we give an example of an orbifold, namely the $\mathbb{Z}_2$ orbifold theory for compactified bosons.

1 Introduction

The aim of this text is to study conformal field theories on the torus and investigate the implications of modular invariance on them. In section 2, the modular group and its generators are presented, giving an overview of modular transformations which are a key tool to derive partition functions on the torus. Sections 3.1 & 3.2 describe the torus in general and the partition function which is the object of interest in our models. Next, different examples for conformal field theories on the torus and their modular invariant partition functions are given. We work out the free boson partition function in detail, a calculation which involves ζ regularization which is a very important technique in both theoretical physics and number theory. We give a sketch of the study of the free fermion and compactified boson and take a look at the requirements following from modular invariance for a multi-component model. The last section is an introduction to the $\mathbb{Z}_2$ orbifold theory which tries to "mod" an inversion symmetry out of the compactified boson while keeping modular invariance.

This text is part of the proseminar *Conformal field theory & String theory* taking place in the spring semester of 2013 at ETH Zurich and is a detailed summary of a talk held by myself in March 2013. It is mostly based on the presentation given in chapter 10 of [1], but some calculations have been added and worked out in more detail in order to make it easier to follow. The main focus lies on the calculation of the free boson partition function on the torus, and we give the other examples as a sketch of derivation.

2 The modular group

2.1 Modular transformations

Definition 1

The modular group Γ is the group of all linear fractional transformations of the upper half complex plane $\mathbb{H}$ of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z}, ad-bc=1.$$

All of the above transformations keep z in $\mathbb{H}$. We can identify the modular transformation described by the numbers (a,b,c,d) with a 2x2 matrix with integer entries, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The composition of functions corresponds to matrix multiplication, as we can easily check:

Let $f : \mathbb{H} \to \mathbb{H}, z \mapsto \frac{az+b}{cz+d}$ and $g : \mathbb{H} \to \mathbb{H}, z \mapsto \frac{ez+f}{gz+h}$. Then, we have

$$\begin{aligned} (g \circ f)(z) = g(f(z)) = g\left(\frac{az+b}{cz+d}\right) &= \frac{e\left(\frac{az+b}{cz+d}\right)+f}{g\left(\frac{az+b}{cz+d}\right)+h} \\ = \frac{cz+d}{cz+d}\left(\frac{aez+be+fcz+df}{agz+bg+hcz+hd}\right) &= \frac{(ae+cf)z+(be+df)}{(ag+ch)z+(bg+dh)}. \end{aligned} \tag{1}$$

This is exactly the modular transformation described by

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}, \tag{2}$$

so we see that the matrix multiplication corresponds to composition of functions. We also see that the group operation is associative.

The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, which also has integer entries and the same determinant, and the identity is described by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Because $ad - bc = 1$, all modular transformations will be described by special linear matrices in $\mathrm{SL}_2(\mathbb{Z}) = \{\, A \in \mathrm{Mat}\,(2 \times 2, \mathbb{Z}) |\, \det A = 1\}$. However, this is too much, because a change of all four signs of a, b, c, d does not change the modular transformation. Therefore, the matrices describing modular transformations are only determined up to a sign. Formally, this means that we have to take the quotient of $\mathrm{SL}_2(\mathbb{Z})$ by its center, which is $\{\mathbb{1}, -\mathbb{1}\}$.

That way, we end up at the matrix group describing modular transformations, which is $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\,\{\mathbb{1}, -\mathbb{1}\}$. We call $\mathrm{PSL}_2(\mathbb{Z})$ the *projective special linear group*. Our original Γ is of course isomorphic to it.

2.2 Generators of the modular group

Definition 2

> *Let $\mathcal{T} : \mathbb{H} \to \mathbb{H}, z \mapsto z + 1$ (unit translation to the right), and*
> *$\mathcal{S} : \mathbb{H} \to \mathbb{H}, z \mapsto -\frac{1}{z}$ (inversion in the unit circle, followed by reflection about Re $z = 0$).*

$\mathcal{T}$ and $\mathcal{S}$ can be shown to be the generators of the modular group. This means that we can write any modular transformation as a combination of powers of $\mathcal{T}$ and $\mathcal{S}$. The matrices associated with the generators are for instance

$$\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

The generators also have the defining properties $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = \mathbb{1}$. This enables us to give a presentation of the modular group:

$$\Gamma \cong \langle \mathcal{S}, \mathcal{T} |\, \mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = \mathbb{1} \rangle. \tag{4}$$

This shows that Γ is isomorphic to the free product of two cyclic groups, C_2 and C_3. However, this is not of interest here.

2.3 Special functions with regard to the modular group

When investigating modular transformations, there are some functions which occur very often and are connected to elliptic functions. In order to give an idea of the functions with which we will be dealing later, we start with a short introduction to theta functions and the Dedekind η function, including the definitions and their modular properties.

2.3.1 Theta functions

Theta functions are special functions of several complex variables. They arise for instance as solutions of the heat equation. They are quasiperiodic which also makes them important in the theory of elliptic functions. Another well-known example of their occurrence is Bernhard Riemann's proof of the functional equation for the Riemann ζ function [2].

Let us define the 3 theta functions which are of interest in our case. We will go directly to the relevant theta functions of $\tau \in \mathbb{H}$, instead of considering two variables $(z, \tau) \in \mathbb{C} \times \mathbb{H}$. We give both the series and product identities without proof.

Definition 3

Let $\tau \in \mathbb{H}$, $q = e^{2\pi i \tau}$. Then define

$$\Theta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2/2} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2,$$
$$\Theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty} (1-q^n)\left(1+q^{n-\frac{1}{2}}\right)^2,$$
$$\Theta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty} (1-q^n)\left(1-q^{n-\frac{1}{2}}\right)^2.$$

Above theta functions are holomorphic on $\mathbb{H}$ and have rather simple transformation properties under the modular group, as we can see in table (1).

2.3.2 Dedekind's η function

Definition 4

Let $\tau \in \mathbb{H}$, $q = e^{2\pi i \tau}$. Dedekind's η function is then defined to be

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n).$$

The η function is holomorphic on $\mathbb{H}$ and cannot be extended analytically to a bigger domain. Its connection to the theta functions is very simple:

$$\eta^3(\tau) = \frac{1}{2}\Theta_2(\tau)\Theta_3(\tau)\Theta_4(\tau). \tag{5}$$

2.3.3 Table of modular properties

The theta functions, as well as Dedekind's η function, are not invariant under modular transformations; however, they transform in a simple way. Table (1) shows the important relations.

$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau)$	$\Theta_2(\tau+1) = e^{\frac{\pi i}{4}}\Theta_2(\tau)$
$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$	$\Theta_2\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\Theta_4(\tau)$
$\Theta_3(\tau+1) = \Theta_4(\tau)$	$\Theta_4(\tau+1) = \Theta_3(\tau)$
$\Theta_3\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\Theta_3(\tau)$	$\Theta_4\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\Theta_2(\tau)$

Table 1: Behaviour of some special functions under modular transformations.

3 Conformal field theory on the torus

3.1 The torus

We can define a torus to be a parallelogram whose opposite edges are identified with each other. It could also be seen as a cylinder whose ends were glued together. Let us look at the parallelogram: We specify two linearly independent *lattice vectors* and identify points which differ by an integer combination of these vectors. On the complex plane, we can write the vectors as two complex numbers ω_1, ω_2 which are called *periods* of the lattice. We call $\tau = \frac{\omega_2}{\omega_1}$ the *modular parameter*, which can be chosen so that $\tau \in \mathbb{H}$. We write the lattice defined by the vectors ω_1, ω_2 as $L(\omega_1, \omega_2)$.

Now, the set $\mathbb{C}/L(\omega_1, \omega_2)$ is homeomorphic to the torus. The complex structure of $\mathbb{C}$ induces a complex structure on $\mathbb{C}/L(\omega_1, \omega_2)$. So, we can say that the pair of lattice vectors (ω_1, ω_2) defines a complex structure on the torus. Two pairs of lattice vectors, (ω_1, ω_2) and (ω_1', ω_2'), define the same complex structure on the torus if $\tau = \frac{\omega_2}{\omega_1}$ and $\tau' = \frac{\omega_2'}{\omega_1'}$ are connected via a modular transformation (a detailed proof of this can be found in [3]).

Our theory on the torus does not depend on the overall scale of our lattice, and the absolute orientation of the periods is not important either. In a geometric reasoning, we can look at the generators of modular transformations: $\mathcal{T}$ corresponds to cutting the torus at a fixed point on the imaginary axis, rotating one piece by 2π and sticking the pieces back together. $\mathcal{S}$ corresponds to looking at the torus from the side. Therefore, modular transformations of τ do not change the theory on the torus, so when investigating conformal field theories on the torus, we want to look for partition functions which are invariant under modular transformations. Let us next look at the partition function in general.

3.2 The partition function

In correspondance with the partition function from statistical mechanics,

$$Z = \mathrm{Tr} e^{-\frac{H}{kT}}, \tag{6}$$

where H is the Hamiltonian, the partition function in quantum field theory embodies the spectrum of our theory. Generally, one can write it in a path integral formulation:

$$Z = \int [d\varphi]\, e^{-\frac{S[\varphi]}{\hbar}}, \tag{7}$$

where $S[\varphi]$ is the action depending on a set of local fields $[\varphi]$.

However, we also want to find a similar expression to (6). First, we need to specify space and time directions on the torus. We will take space to run along the real axis and time along the imaginary axis. Let H be the Hamiltonian generating translations along the time direction and P the total momentum generating translations along the space direction. Then the operator which translates the system parallel to the period ω_2 over a distance a in Euclidean space-time is given by

$$\exp\left(-\frac{a}{|\omega_2|}\left[H \mathrm{Im}\, \omega_2 - iP \mathrm{Re}\, \omega_2\right]\right). \tag{8}$$

Now we will regard a as a lattice spacing, then this operator will take us from one row of a lattice to the next. If the complete period contains m lattice spacings, that is $|\omega_2| = ma$, then we obtain the partition function by taking the trace of the translation operator to the m-th power:

$$Z = \mathrm{Tr}\exp\left(-H \mathrm{Im}\,\omega_2 + iP \mathrm{Re}\,\omega_2\right). \tag{9}$$

In the next step, we will regard the torus as a cylinder of circumference L whose ends have been stuck together and use this to find the partition function in terms of the Virasoro generators L_0 and $\overline{L}_0$. On a cylinder of circumference L, the Hamiltonian and total momentum are given by:

$$H = \frac{2\pi}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right), \quad P = \frac{2\pi i}{L}\left(L_0 - \overline{L}_0\right). \tag{10}$$

In our analogue to the cylinder, ω_1 is real and equal to L, so the final result for the partition function dependent on τ is

$$\begin{aligned} Z(\tau) &= \mathrm{Tr}\exp\left(\pi i\left[(\tau - \overline{\tau})\left(L_0 + \overline{L}_0 - \frac{c}{12}\right) + (\tau + \overline{\tau})\left(L_0 - \overline{L}_0\right)\right]\right) \\ &= \mathrm{Tr}\exp\left(2\pi i\left[\tau\left(L_0 - \frac{c}{24}\right) - \overline{\tau}\left(\overline{L}_0 - \frac{c}{24}\right)\right]\right). \end{aligned} \tag{11}$$

Now we introduce $q = e^{2\pi i\tau}$, $\overline{q} = e^{-2\pi i\overline{\tau}}$ and find

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\overline{q}^{\overline{L}_0 - \frac{c}{24}}\right). \tag{12}$$

This result is connected to the Virasoro characters $\chi_{(c,h)}(\tau) = \mathrm{Tr} q^{L_0 - \frac{c}{24}}$, where c is the central charge and h is the weight, which is the eigenvalue of L_0. We will apply this to the free boson on the torus now.

3.3 The free boson on the torus

We will now investigate the free boson on the torus and calculate its partition function. First, note that we should discard the zero-mode because its contribution to Z is infinite. Then, we note that we can calculate the Virasoro characters to be $\chi_{(c,h)}(\tau) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} = \frac{q^{h + \frac{1-c}{24}}}{\eta(\tau)}$ (a derivation of this can be found in [1], chapter 7). Comparing this with (12), we expect the partition function of the free boson to behave like

$$Z_{bos}(\tau) \propto \frac{1}{|\eta(\tau)|^2}. \tag{13}$$

However, the above expression is not modular invariant. We need a proper multiplicative factor that makes it invariant under modular transformations. We can check that $Z_{bos}(\tau) = \frac{1}{\sqrt{\mathrm{Im}\,\tau}|\eta(\tau)|^2}$ is modular invariant, using the properties given in table (1): Under $\mathcal{S}$, we have

$$\begin{aligned} \frac{1}{\sqrt{\mathrm{Im}\,\left(-\frac{1}{\tau}\right)}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2} &= \left(\sqrt{\frac{-\frac{1}{\tau} + \frac{1}{\overline{\tau}}}{2i}}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2\right)^{-1} \\ &= \left(\sqrt{\frac{-\overline{\tau} + \tau}{2i\,|\tau|^2}}\sqrt{|\tau|^2}\,|\eta(\tau)|^2\right)^{-1} = \frac{1}{\sqrt{\mathrm{Im}\,\tau}\,|\eta(\tau)|^2}. \end{aligned} \tag{14}$$

The invariance under $\mathcal{T}$ is trivial because η just picks up a phase and $\mathrm{Im}\,(\tau + 1) = \mathrm{Im}\,\tau$.

Therefore, this is the result we anticipate for the free-boson partition function on the torus. In the next section, we will carry out the calculation of the functional integral and see that this is indeed the result.

3.3.1 Detailed calculation in the path-integral formalism

This calculation follows Chapter 10.2 in [1], with more detailed additional explanations where appropriate. Discarding the zero-mode, we write the free-boson partition function as [1]:

$$Z_{bos}(\tau) = \int [d\varphi]\, \sqrt{A}\, \delta\left(\int d^2x\ \varphi\varphi_0\right) \exp\left(-\frac{1}{2}\int d^2x\ \left(\nabla\varphi\right)^2\right). \tag{15}$$

The coordinate integrals in above expression are over the torus. $A = \mathrm{Im}\left(\omega_2\omega_1^*\right)$ is the area of the torus, and $\varphi_0 = A^{-\frac{1}{2}}$ is the normalized eigenfunction of the zero-mode. The delta distribution ensures that the zero-mode is discarded: its argument is the coefficient of the zero-mode in any field configuration, and the square root of the area makes the expression dimensionless.

In order to carry out the calculation, let us expand the field φ in terms of the normalized eigenfunctions of the Laplacian ∇^2. We write these eigenfunctions with eigenvalues $-\lambda_n$ as φ_n. For the nonzero modes, we have $\lambda_n \neq 0$ and we can calculate the functional integral:

$$\begin{aligned} Z_{bos}(\tau) &= \sqrt{A}\ \int \prod_i dc_i \exp\left(-\frac{1}{2}\sum_n \lambda_n c_n^2\right) && (16)\\ &= \sqrt{A}\ \prod_n \left(\frac{2\pi}{\lambda_n}\right)^{\frac{1}{2}}. && (17)\end{aligned}$$

However, this product diverges. To assign a finite value it, we use the technique of ζ function regularization. Define

$$G(s) = \sum_{n,\ \lambda_n\neq 0} \frac{1}{\lambda_n^s}. \tag{18}$$

This function is analytic for sufficiently large values of s. There, we have

$$\frac{d}{ds}G(s) = \frac{d}{ds}\sum_n \frac{1}{\lambda_n^s} = \sum_n \frac{d}{ds}\exp\left(-\log\left(\lambda_n\right)s\right) = -\sum_n \log\left(\lambda_n\right)\frac{1}{\lambda_n^s}. \tag{19}$$

There also exists an analytic continuation of G to a domain including $s = 0$. What we will do in the rest of this section is try to find this analytic continuation by playing with the definition and trying to find some identity for $G(s)$ which enables us to find values for $G(s)$ in the region where the sum does not converge. Comparing this with Z_{bos}, we see that up to a factor of $(2\pi)^{\frac{1}{2}}$ for each mode (which does not give us information about the dependence of Z_{bos} on τ), we formally have

$$Z_{bos}(\tau) = \sqrt{A}\ \exp\left(\frac{1}{2}G'(0)\right). \tag{20}$$

In our case, the eigenvalues are labeled by two integers m and n and look as follows [1]:

$$\lambda_{n,m} = (2\pi)^2\left|nk_1 + mk_2\right|^2, \tag{21}$$

with $k_1 = -i\frac{\omega_2}{A}$, $k_2 = i\frac{\omega_1}{A}$. k_1, k_2 are the lattice vectors dual to ω_1, ω_2. The above expression is the norm squared of a lattice vector in the dual lattice. The eigenfunctions would look like $exp(i\langle x, \xi\rangle)$, where ξ ranges over the dual lattice. Therefore, we can see the eigenvalue as a Fourier mode.

Inserting the eigenvalues, we obtain

$$\left|\frac{2\pi\omega_1}{A}\right|^{2s} G(s) = \sum_{(m,n)\neq(0,0)} \frac{1}{|m+n\tau|^{2s}} \tag{22}$$

$$= \sum_m \frac{1}{|m|^{2s}} + \sum_{n\neq 0}\left(\sum_m \frac{1}{|m+n\tau|^{2s}}\right) \tag{23}$$

$$= 2\zeta(2s) + \sum_{n\neq 0}\left(\sum_m \frac{1}{|m+n\tau|^{2s}}\right), \tag{24}$$

where $\zeta(z)$ is the Riemann ζ function. As we sum over all values of m in the second term, it is periodic in $n\tau$ with period 1. Therefore, we will now calculate its Fourier series, writing $\tau = \tau_1 + i\tau_2$:

$$\sum_m \frac{1}{|m+n\tau|^{2s}} = \sum_p e^{2\pi i p n \tau_1} \int_0^1 dy\, e^{-2\pi i p y} \sum_m \frac{1}{\left[(m+y)^2 + n^2\tau_2^2\right]^s} \tag{25}$$

$$= \sum_p e^{2\pi i p n \tau_1} \sum_m \int_m^{m+1} dy\, e^{-2\pi i p y} \frac{1}{\left[y^2 + n^2\tau_2^2\right]^s} \tag{26}$$

$$= \sum_p \int_{-\infty}^{\infty} dy\, e^{2\pi i p (n\tau_1 - y)} \frac{1}{\left[y^2 + n^2\tau_2^2\right]^s} \tag{27}$$

$$= \frac{1}{\Gamma(s)} \sum_p \int_{-\infty}^{\infty} dy\, e^{2\pi i p (n\tau_1 - y)} \int_0^{\infty} dt\, t^{s-1} e^{-t(y^2 + n^2\tau_2^2)} \tag{28}$$

$$= \frac{\sqrt{\pi}}{\Gamma(s)} \sum_p \int_0^{\infty} dt\, t^{s-\frac{3}{2}} e^{-\left[t n^2 \tau_2^2 + \frac{\pi^2 p^2}{t} - 2\pi i p n \tau_1\right]}. \tag{29}$$

In the fourth line of this calculation, Euler's Γ function was used:

$$\Gamma(s) = \int_0^{\infty} dx\, x^{s-1} e^{-x}. \tag{30}$$

After the substitution $zt = x$, the following identity is obtained:

$$\frac{1}{z^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} e^{-zt}. \tag{31}$$

This identity was used with $z = y^2 + n^2\tau_2^2$.

In the last step, we completed the square in the integral over dy and substituted $i\sqrt{t}y - \frac{\pi p}{\sqrt{t}} = i\sqrt{t}y'$.

$$\int_{-\infty}^{\infty} dy\, e^{-2\pi i p y - t y^2} = \int_{-\infty}^{\infty} dy\, e^{\left(i\sqrt{t}y - \frac{\pi p}{\sqrt{t}}\right)^2 - \frac{\pi^2 p^2}{t}} \tag{32}$$

$$= \int_{-\infty}^{\infty} dy'\, e^{-t y'^2 + \frac{\pi^2 p^2}{t}} = \sqrt{\frac{\pi}{t}} e^{-\frac{\pi^2 p^2}{t}}. \tag{33}$$

Now let us take a look at the integral 29 for $p = 0$. Using (31) again, we find that in this case the integral gives

$$\int_0^{\infty} dt\, t^{s-\frac{3}{2}} e^{-t n^2 \tau_2^2} = \Gamma\left(s - \frac{1}{2}\right) |n\tau_2|^{1-2s}. \tag{34}$$

If we plug this back in (24), this term will be summed over all $n \neq 0$, which gives us

$$\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\sum_{n\neq 0}|n\tau_2|^{1-2s} = 2\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}|\tau_2|^{1-2s}\zeta(2s-1). \tag{35}$$

Now we can use the functional equation of the Riemann ζ function:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{36}$$

Alltogether, with the substitution $t = t'\frac{\pi p}{n\tau_2}$, we find for our ζ-like $G(s)$:

$$\begin{aligned}\Gamma(s)\left(\frac{\tau_2}{\pi}\right)^{s-\frac{1}{2}}&\left|\frac{2\pi\omega_1}{A}\right|^{2s}G(s)\\ &=2\Gamma(s)\zeta(2s)\left(\frac{\tau_2}{\pi}\right)^{s-\frac{1}{2}}+2\Gamma(1-s)\zeta(2-2s)\left(\frac{\tau_2}{\pi}\right)^{\frac{1}{2}-s}\\ &\quad+\sqrt{\pi}\sum_{p\neq 0}\sum_{n\neq 0}e^{2\pi i p n\tau_1}\int_0^\infty\frac{dt}{t}t^{s-\frac{1}{2}}\left|\frac{p}{n}\right|^{s-\frac{1}{2}}e^{-\pi|np|\tau_2\left(t+\frac{1}{t}\right)}.\end{aligned} \tag{37}$$

The above is well-defined for $s > 1$ and gives us a different way of writing $G(s)$ there. However, it has the major advantage that it is symmetric under $s \mapsto 1-s$ which enables us to find values for $G(0)$ and $G'(0)$. Let us expand $G(s)$ around $s = 0$ up to first order. We know from complex analysis that Γ has a simple pole with residue 1 at the origin, that is $\Gamma(s) \sim \frac{1}{s}$, so we only need to calculate the integral at $s = 0$:

$$\int_0^\infty\frac{dt}{t}t^{-\frac{1}{2}}e^{-\pi|np|\tau_2\left(t+\frac{1}{t}\right)} = \frac{1}{|pn|^{\frac{1}{2}}}e^{-2\pi|pn|\tau_2}. \tag{38}$$

Furthermore, $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(0) = -\frac{1}{2}$, which gives us

$$G(s) = -1 - 2s\log\left|\frac{A}{\omega_1}\right| + \frac{1}{3}s\pi\tau_2 + s\sum_{p\neq 0}\sum_{n\neq 0}\frac{1}{|p|}e^{2\pi i p n\tau_1 - 2\pi|pn|\tau_2} + \mathcal{O}(s^2). \tag{39}$$

Now we want to obtain control over the double sum. We use our already introduced $q = e^{2\pi i\tau}$ to write it:

$$\begin{aligned}\sum_{p\neq 0}\sum_{n\neq 0}\frac{1}{|p|}e^{2\pi i p n\tau_1-2\pi|pn|\tau_2} &= \sum_{p,n>0}\frac{2}{p}\left(e^{2\pi i n p\tau_1-2\pi n p\tau_2}+e^{-2\pi i n p\tau_1-2\pi n p\tau_2}\right)\\ &= \sum_{p,n>0}\frac{2}{p}\left(q^{np}+\overline{q}^{np}\right)\\ &= -2\sum_{n>0}\left(\log\left(1-q^n\right)+\log\left(1-\overline{q}^n\right)\right)\\ &= -2\log|\eta(\tau)|^2 - \frac{1}{3}\pi\tau_2.\end{aligned} \tag{40}$$

We have used the fact that $\log(1-x) = -\sum_{k=1}^\infty\frac{x^k}{k}$ for $|x| \leq 1, x \neq 1$. This we can use because $\tau \in \mathbb{H}$, and therefore $|q| = e^{-2\pi\tau_2} < 1$. We also have $q \neq 1$ because $\tau \notin \mathbb{Z}$.

Now $\frac{\sqrt{A}}{|\omega_1|} = \sqrt{\tau_2}$, and we have the result

$$G'(0) = -2\log\left(\sqrt{A\tau_2}\,|\eta(\tau)|^2\right), \tag{41}$$

giving the final free-boson partition function

$$Z_{bos}(\tau) = \sqrt{A}\exp\left(\frac{1}{2}G'(0)\right) = \frac{1}{\sqrt{\operatorname{Im}\tau}\,|\eta(\tau)|^2}. \tag{42}$$

This is exactly what we anticipated earlier.

3.4 The free fermion on the torus

The free-fermion action is

$$S = \frac{1}{2\pi}\int d^2x\,\left(\overline{\psi}\partial\overline{\psi} + \psi\overline{\partial}\psi\right). \tag{43}$$

Since the two fields ψ and $\overline{\psi}$ are decoupled, the integral can be calculated to be the Pfaffian determinant of the differential operators appearing in it, ∂ and $\overline{\partial}$ (see [1], chapter 2):

$$Z = \operatorname{Pf}(\partial)\operatorname{Pf}(\partial) = \sqrt{\det\nabla^2}. \tag{44}$$

Now we want to impose periodicity conditions on the fermions, which in the end affects the eigenvalues of the Laplacian and therefore the determinant. We assume that the fermions pick up a phase when being translated by a period:

$$\psi(z+\omega_1) = e^{2\pi i v}\psi(z), \quad \psi(z+\omega_2) = e^{2\pi i u}\psi(z). \tag{45}$$

However, the demand that the action must be periodic when $z \mapsto z+\omega_1$ or $z \mapsto \omega_2$ restricts the possible values of (v,u). The fields can at most pick up a sign and there are only four possible configurations:

$$\begin{aligned}
(v,u) &= (0,0) && \text{(R,R)},\\
(v,u) &= \left(0,\frac{1}{2}\right) && \text{(R,NS)},\\
(v,u) &= \left(\frac{1}{2},0\right) && \text{(NS,R)},\\
(v,u) &= \left(\frac{1}{2},\frac{1}{2}\right) && \text{(NS,NS)}.
\end{aligned} \tag{46}$$

We call the periodic boundary conditions Ramond and the antiperiodic ones Neveu-Schwarz. A set of periodicity conditions (v,u) like above is called a *spin structure* for the fermion.

Let us calculate the partition functions $Z_{v,u}$ associated with the boundary conditions. We can consider the partition function obtained by just integrating the holomorphic field ψ, because ψ and $\overline{\psi}$ are decoupled. We call such a partition function $d_{v,u}$ and have

$$Z_{v,u} = |d_{v,u}|^2. \tag{47}$$

When carrying out a careful implementation of the periodicity conditions, the Fermion number F so that $(-1)^F$ anticommutes with $\psi(z)$ will arise:

$$F = \sum_{k\geq 0} F_k \quad F_k = b_{-k}b_k. \tag{48}$$

This is needed to set time-periodic boundary conditions as the natural choice in the time direction are anti-periodic boundary conditions: The correlation function of an odd number of fermions vanishes, so we consider a time-ordered path integral with an even number of fermionic fields. Then, making a loop in time with a field amounts to passing it through all the other fields, which generates an overall minus sign. Therefore, in the time-periodic case, the time-evolution operator is multiplied with $(-1)^F$, which modifies the partition functions:

$$\begin{aligned} d_{0,0} &= \frac{1}{\sqrt{2}} \mathrm{Tr}(-1)^F q^{L_0 - \frac{1}{48}}, \\ d_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}} \mathrm{Tr} q^{L_0 - \frac{1}{48}}, \\ d_{\frac{1}{2},0} &= \mathrm{Tr}(-1)^F q^{L_0 - \frac{1}{48}}, \\ d_{\frac{1}{2},\frac{1}{2}} &= \mathrm{Tr} q^{L_0 - \frac{1}{48}}. \end{aligned} \tag{49}$$

The factors of $\sqrt{2}$ in the space-periodic cases are conventional and simplify the transformation of the partition functions under modular transformations, as we will see. When calculating the traces, we use ([1], chapter 6):

$$\begin{aligned} L_0 &= \sum_{k>0} k b_{-k} b_k && \left(\mathrm{NS} : k \in \mathbb{Z} + \frac{1}{2}\right) \\ &= \sum_{k>0} k b_{-k} b_k + \frac{1}{16} && (\mathrm{R} : k \in \mathbb{Z}) . \end{aligned} \tag{50}$$

As $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$ (for operators A, B acting on different factors of a tensor product), we can split the trace of the product which appears into a product of traces of the form $\mathrm{Tr} q^{k b_{-k} b_k}$. Furthermore, for a given fermion mode, there are only two states which makes the traces:

$$\begin{aligned} \mathrm{Tr} q^{k b_{-k} b_k} &= 1 + q^k, \\ \mathrm{Tr} q^{k b_{-k} b_k} (-1)^F &= 1 - q^k. \end{aligned} \tag{51}$$

Alltogether, we obtain the following partition functions:

$$\begin{aligned} d_{0,0} &= 0, \\ d_{0,\frac{1}{2}} &= \sqrt{\frac{\Theta_2(\tau)}{\eta(\tau)}}, \\ d_{\frac{1}{2},0} &= \sqrt{\frac{\Theta_4(\tau)}{\eta(\tau)}}, \\ d_{\frac{1}{2},\frac{1}{2}} &= \sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}}. \end{aligned} \tag{52}$$

Now we try to construct a modular invariant partition function from this. We can check the modular properties of above partition functions using table (1), which are given in table (2).

$d_{0,\frac{1}{2}}(\tau+1) = e^{\frac{\pi i}{8}} d_{0,\frac{1}{2}}(\tau)$	$d_{0,\frac{1}{2}}\left(-\frac{1}{\tau}\right) = d_{\frac{1}{2},0}(\tau)$
$d_{\frac{1}{2},0}(\tau+1) = e^{-\frac{\pi i}{24}} d_{\frac{1}{2},\frac{1}{2}}(\tau)$	$d_{\frac{1}{2},0}\left(-\frac{1}{\tau}\right) = d_{0,\frac{1}{2}}(\tau)$
$d_{\frac{1}{2},\frac{1}{2}}(\tau+1) = e^{-\frac{\pi i}{24}} d_{\frac{1}{2},0}(\tau)$	$d_{\frac{1}{2},\frac{1}{2}}\left(-\frac{1}{\tau}\right) = d_{\frac{1}{2},\frac{1}{2}}(\tau)$

Table 2: Modular properties of the holomorphic partition functions appearing in the free boson model.

The phase factors are not a problem since the full partition functions are the modulus squared of the holomorphic ones. However, the fact that the partition functions mix under modular transformations tells us that there are only two ways of constructing a modular invariant partition function: The first one are the periodic (R,R) conditions, but the partition function vanishes there; the second one is including all the other three possibilities for the boundary conditions. So modular invariance restricts us: all the three conformal fields associated with a fermion central charge of $c = \frac{1}{2}$ have to be present in the theory. Then, we have

$$\begin{aligned} Z &= Z_{\frac{1}{2},\frac{1}{2}} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},\frac{1}{2}} \\ &= \left|\frac{\Theta_2}{\eta}\right| + \left|\frac{\Theta_3}{\eta}\right| + \left|\frac{\Theta_4}{\eta}\right|. \end{aligned} \tag{53}$$

We note that this is just twice the partition function of the Ising model on a torus.

3.5 The compactified boson

Let us take a look at another example for a conformal field theory on the torus:

This time, we want to consider a winding occurring when we go from the point z to the point $z + \omega_1$ or $z + \omega_2$, which are equivalent. This gives the following boundary condition:

$$\varphi(z + k\omega_1 + k'\omega_2) = \varphi(z) + 2\pi R(km + k'm') \quad k, k' \in \mathbb{Z}. \tag{54}$$

A pair of integers (m, m') then specifies a class of configurations which obey the above periodicity condition, and the assigned partition function $Z_{m,m'}$ is found by integrating over the configurations of this class. When integrating, we want to decompose φ into a periodic ("free") field $\tilde{\varphi}$ and a classical solution to the equation of motion which we denote by $\varphi^{cl}_{m,m'}$. $\varphi^{cl}_{m,m'}$ has a vanishing Laplacian. Our decomposition is then

$$\begin{aligned} \varphi &= \varphi^{cl}_{m,m'} + \tilde{\varphi}, \\ \varphi^{cl}_{m,m'} &= 2\pi R\left[\frac{z}{\omega_1}\frac{m\overline{\tau} - m'}{\overline{\tau} - \tau} - \frac{\overline{z}}{\omega_1^*}\right]. \end{aligned} \tag{55}$$

Because $\varphi^{cl}_{m,m'}$ has vanishing Laplacian, we have

$$\int d^2x \nabla\varphi^{cl}_{m,m'} \nabla\tilde{\varphi} = -\int d^2x \tilde{\varphi}\Delta\varphi^{cl}_{m,m'} = 0. \tag{56}$$

Therefore, $S[\varphi] = S[\tilde{\varphi}] + S[\varphi^{cl}_{m,m'}]$. The second term is

$$\begin{aligned} S[\varphi^{cl}_{m,m'}] &= \frac{1}{2\pi}\int dz d\overline{z} \partial\varphi^{cl}_{m,m'} \overline{\partial}\varphi^{cl}_{m,m'} \\ &= \pi R^2 \frac{|m\tau - m'|^2}{2\text{Im }\tau}, \end{aligned} \tag{57}$$

and the functional integration over the periodic field gives, of course, Z_{bos}. This leads to the following partition function:

$$Z_{m,m'}(\tau) = Z_{bos}(\tau) \exp\left[-\frac{\pi R^2 \left|m\tau - m'\right|^2}{2\text{Im }\tau}\right]. \tag{58}$$

Now we want to check the modular properties of this partition function and see if we can construct a modular invariant one:

$$\begin{aligned} Z_{m,m'}(\tau+1) &= Z_{bos}(\tau) \exp\left[-\pi R^2 \frac{\left|m(\tau+1) - m'\right|^2}{2\text{Im }\tau}\right] \\ &= Z_{bos}(\tau) \exp\left[-\pi R^2 \frac{\left|m\tau - (m'-m)\right|^2}{2\text{Im }\tau}\right] \\ &= Z_{m,m'-m}(\tau). \end{aligned} \tag{59}$$

$$\begin{aligned} Z_{m,m'}\left(-\frac{1}{\tau}\right) &= Z_{bos}(\tau) \exp\left[-\pi R^2 \frac{\left|m\left(\frac{1}{\tau}\right) - m'\right|^2}{2\text{Im }\frac{1}{\tau}}\right] \\ &= Z_{bos}(\tau) \exp\left[-\pi R^2 \frac{\left|-m'\tau - m\right|^2}{2\text{Im }\tau}\right] \\ &= Z_{-m',m}(\tau). \end{aligned} \tag{60}$$

Therefore, we see that the sum of the partition functions over all (m, m') with equal weights is modular invariant. This gives us the partition function

$$Z(R) = \frac{R}{\sqrt{2}} Z_{bos}(\tau) \sum_{m,m'} \exp\left[-\frac{\pi R^2 \left|m\tau - m'\right|^2}{2\text{Im }\tau}\right], \tag{61}$$

which we still want to rewrite. Using Poisson's resummation formula, we find

$$Z(R) = \frac{1}{\left|\eta(\tau)\right|^2} \sum_{e,m\in\mathbb{Z}} q^{\left(\frac{e}{R}+\frac{mR}{2}\right)^2/2} \overline{q}^{\left(\frac{e}{R}-\frac{mR}{2}\right)^2/2}. \tag{62}$$

We can see this as the sum over all possible electric charges of vertex operators and all possible windung numbers (which correspond to magnetic charges) of the $c = 1$ Virasoro characters squared with conformal dimension

$$h_{e,m} = \frac{1}{2}\left(\frac{e}{R} + \frac{mR}{2}\right)^2, \quad \overline{h}_{e,m} = \frac{1}{2}\left(\frac{e}{R} - \frac{mR}{2}\right)^2. \tag{63}$$

It is also remarkable that this model has an electric-magnetic duality which leads to the partition function being invariant under the interchange $R \leftrightarrow \frac{2}{R}$.

3.6 An assembly of bosons

As a next step, we want to see how we can form modular invariant partition functions out of an assembly of compactified free bosons. For this, we need a new concept, the concept of a multidimensional lattice.

Definition 5

An n-dimensional lattice Γ is a set of points in $\mathbb{R}^n$ with the property that its elements can be written as an integer linear combination of a set of n basis vectors ϵ_i:

$$\Gamma = \left\{ x = \sum_i x_i \epsilon_i \in \mathbb{R}^n \middle| x_i \in \mathbb{Z} \right\}. \tag{64}$$

Such a lattice is called Lorentzian with signature (r, s) if it possesses through $\mathbb{R}^n$ an indefinite inner product $\operatorname{diag}(1, ..., 1, -1, ..., -1)$ with r positive signs and s negative signs. We write $\langle x, y\rangle$ for the inner product of two elements x, y of the lattice.

To each lattice Γ, there is a dual lattice $\Gamma^* = \{p \in \mathbb{R}^n | \langle x, p\rangle \in \mathbb{Z}\}$. In the case $\Gamma = \Gamma^*$, the lattice is called self-dual. If the property $\langle x, y\rangle \in \mathbb{Z}$ holds for all $x, y \in \Gamma$, then the lattice is an integer lattice. Furthermore, if all elements $x \in \Gamma$ have even squared norm, that is $\langle x, x\rangle \in 2\mathbb{Z}$, we say the lattice is even-integer.

3.6.1 Multi-component chiral boson

Now we go back to the partition function given in (62) and introduce $p = \frac{e}{R} + \frac{mR}{2}$, $\overline{p} = \frac{e}{R} - \frac{mR}{2}$ to write

$$Z(R) = \frac{1}{|\eta(\tau)|^2} \sum_{p,\overline{p}} e^{\pi i \left(\tau p^2 - \overline{\tau}\, \overline{p}^2\right)}. \tag{65}$$

The sum is still to be taken over all integer values of e and m. We can define the basis vectors $\epsilon_1 = \left(\frac{1}{R}, \frac{1}{R}\right)$ and $\epsilon_2 = \left(\frac{R}{2}, -\frac{R}{2}\right)$. Using an inner product with signature $(1, 1)$, we see that the set of points $(p, \overline{p})$ forms an even, self-dual, Lorentzian integer lattice. This is not a fluke - in fact, this is related to modular invariance, as we will show now.

Consider, in general, a set of n bosons of which we keep only the holomorphic modes, and a distinct set of $\overline{n}$ bosons of which we keep only the antiholomorphic modes. This theory can be defined by the following expression for the Virasoro generators:

$$\begin{aligned} L_0 &= \frac{1}{2} p^2 + \sum_{i=1}^{n} \sum_{k>0} a_{-k}^{(i)} a_k^{(i)}, \\ \overline{L}_0 &= \frac{1}{2} \overline{p}^2 + \sum_{i=1}^{\overline{n}} \sum_{k>0} \overline{a}_{-k}^{(i)} \overline{a}_k^{(i)}, \end{aligned} \tag{66}$$

where p belongs to some lattice Γ and $\overline{p}$ belongs to some lattice $\overline{\Gamma}$. The partition function of such a system is

$$Z_\Gamma(\tau) = \frac{1}{\eta(\tau)^n \overline{\eta}(\tau)^{\overline{n}}} \sum_{p \in \Gamma, \overline{p} \in \overline{\Gamma}} e^{\pi i \left(\tau p^2 - \overline{\tau}\, \overline{p}^2\right)}. \tag{67}$$

Now we want to find out under what conditions on the lattice and $(n, \overline{n})$ this partition function will be modular invariant. Let us start with the action of $\mathcal{T}$ on Z_Γ:

$$\begin{aligned} Z_\Gamma(\tau + 1) &= \frac{1}{\eta(\tau)^n \overline{\eta}(\tau)^{\overline{n}}} \exp^{2\pi i \frac{(n - \overline{n})}{24}} \sum_{p \in \Gamma, \overline{p} \in \overline{\Gamma}} e^{\pi i \left(\tau p^2 - \overline{\tau}\, \overline{p}^2\right)} e^{\pi i \left(p^2 - \overline{p}^2\right)} \\ &= Z_\Gamma(\tau) e^{2\pi i \frac{(n - \overline{n})}{24}} e^{\pi i \left(p^2 - \overline{p}^2\right)}. \end{aligned} \tag{68}$$

Therefore, we need that $p^2 - \overline{p}^2$ is always an even integer, which means that the lattice $\Gamma \oplus \overline{\Gamma}$ has to be even-integer. Furthermore, we need $(n - \overline{n}) = 0 \bmod 24$.

The investigation of $\mathcal{S}$ needs a generalization of the Poisson resummation formula in the calculation, which will not be worked out here, but the result is that the partition function is invariant under $\mathcal{S}$ if $\Gamma = \Gamma^*$, which means that the lattice is self-dual. Alltogether, we obtain the result that a model built from n holomorphic and $\overline{n}$ antiholomorphic bosons is modular invariant if the so-called charge lattice (the lattice of charge vectors $(p, \overline{p})$) is an even-integer self-dual lattice, and $(n - \overline{n}) = 0 \bmod 24$.

The issue of modular invariance of a multicomponent boson system is important for another topic in this proseminar series, the compactification of the bosonic string.

4 Orbifolds

Now we want to take a look at a variation of the compactified free boson theory. Let us first introduce the term "orbifold", which we will define as in [4]. We can describe it as a generalization of a manifold allowing a discrete set of singular points.

Definition 6

Let $\mathcal{M}$ be a manifold with a discrete group action $\mathcal{G} : \mathcal{M} \to \mathcal{M}$. Then we construct the orbifold $\mathcal{M}/\mathcal{G}$ by identifying points under the equivalence relation $x \sim gx$ for all $g \in \mathcal{G}$.

We say that $\mathcal{G}$ possesses a fixed point $x \in \mathcal{M}$ if for $g \in \mathcal{G}, g \neq \mathbb{1}$, we have $gx = x$.

From this definition, we see that if $\mathcal{G}$ acts freely (which means that it has no fixed points), $\mathcal{M}/\mathcal{G}$ is just a standard manifold. However, if $\mathcal{G}$ has fixed points, the points associated with it will have discrete identifications of their tangent spaces and give rise to singular points.

An example of an orbifold is the $\mathcal{S}_1/\mathbb{Z}_2$ orbifold, which we will discuss. Take $\mathcal{M} = \mathcal{S}_1$ to be the circle, whose coordinate parametrization is $x \equiv x + 2\pi r$, and let $\mathcal{G} = \mathbb{Z}_2 : \mathcal{S}_1 \to \mathcal{S}_1$, which is the action generated by $g : x \mapsto -x$. The fixed points of $\mathbb{Z}_2$ are $x = 0$ and $x = \pi r$. Figure (4) gives a geometric picture, and we see that this orbifold is topologically a line segment.

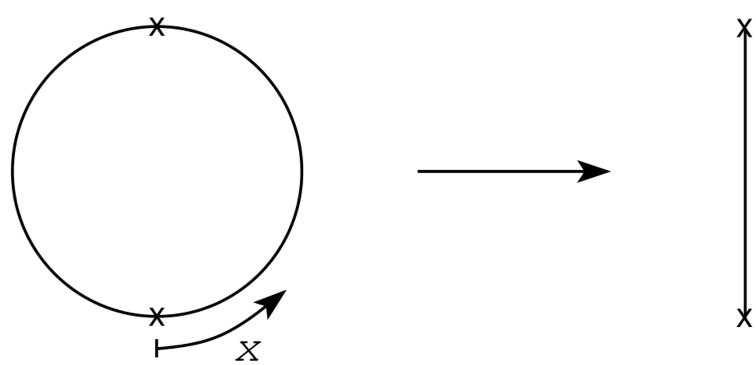

Figure 1: The $\mathcal{S}_1/\mathbb{Z}_2$ orbifold [4].

The application of orbifolds to conformal field theory is usually that we take a modular invariant theory $\mathcal{T}$, the Hilbert space of which allows a discrete symmetry $\mathcal{G}$ and construct a theory $\mathcal{T}/\mathcal{G}$

where the symmetry has been modded out, but the theory is still modular invariant. We will do this with the $\mathbb{Z}_2$ action on the compactified free boson.

4.1 The $\mathbb{Z}_2$ orbifold theory of the compactified free boson

We consider a variation of the compactified free boson theory: Take the $\mathbb{Z}_2$ action on the compactified free boson, that is, we assume that φ does not take it values on the full circle, but on the $\mathcal{S}_1/\mathbb{Z}_2$ orbifold, where the natural action of $\mathbb{Z}_2$ is generated by identifying the angle φ with $-\varphi$. Then, the field can be "twisted" when taken across a period ω_1 or ω_2, which means we have to consider a general boundary condition of the following:

$$\varphi(z + k\omega_1 + l\omega_2) = e^{2\pi i(kv+lu)}\varphi(z). \tag{69}$$

This reminds us of the boundary conditions of the fermion, but this time, the boundary conditions are due to the topology of the space on which our field lives, while in the fermion case, they were due to the fermionic nature of the fields.

The action S for the free boson is invariant under the interchange $\varphi \mapsto -\varphi$, so we could just calculate the integral over half the range of φ as opposed to the circle. Another way of finding the partition function works in analogy to the free fermion: Let the traces over the holomorphic modes be called $f_{v,u}$. The associated partition functions $Z_{v,u}$ are then $|f_{v,u}|^2$, just as before. We have $Z_{0,0} = Z(R)$ for the untwisted sector. Similarly to the free fermion, we insert an operator G which maps φ to $-\varphi$ and anticommutes with φ into the trace in the time-antiperiodic case. This gives us the holomorphic partition functions:

$$\begin{aligned} f_{0,\frac{1}{2}} &= \mathrm{Tr} G q^{L_0 - \frac{1}{24}} = 2\sqrt{\frac{\eta(\tau)}{\Theta_2(\tau)}}, \\ f_{\frac{1}{2},0} &= \mathrm{Tr} q^{L_0 - \frac{1}{48}} = 2\sqrt{\frac{\eta(\tau)}{\Theta_4(\tau)}}, \\ f_{\frac{1}{2},\frac{1}{2}} &= \mathrm{Tr} G q^{L_0 - \frac{1}{48}} = 2\sqrt{\frac{\eta(\tau)}{\Theta_3(\tau)}}. \end{aligned} \tag{70}$$

In order to find a modular invariant partition function built out of these, we check again the modular properties of above holomorphic partition functions. Just like in the case of the free fermion, they mix under modular transformations and we conclude that the only modular-invariant combinations are $Z_{0,0} = Z(R)$ and

$$\left|f_{0,\frac{1}{2}}\right|^2 + \left|f_{\frac{1}{2},0}\right|^2 + \left|f_{\frac{1}{2},\frac{1}{2}}\right|^2. \tag{71}$$

After summing over all types of boundary conditions and projecting on G-invariant states, we obtain the final orbifold partition function [1],[4]:

$$Z_{orb}(R) = \frac{1}{2}\left(Z(R) + \frac{|\Theta_2\Theta_3|}{|\eta|^2} + \frac{|\Theta_2\Theta_4|}{|\eta|^2} + \frac{|\Theta_3\Theta_4|}{|\eta|^2}\right). \tag{72}$$

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