Proseminar Theoretical Physics

Operator Product Expansion

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Defining the Cylinder

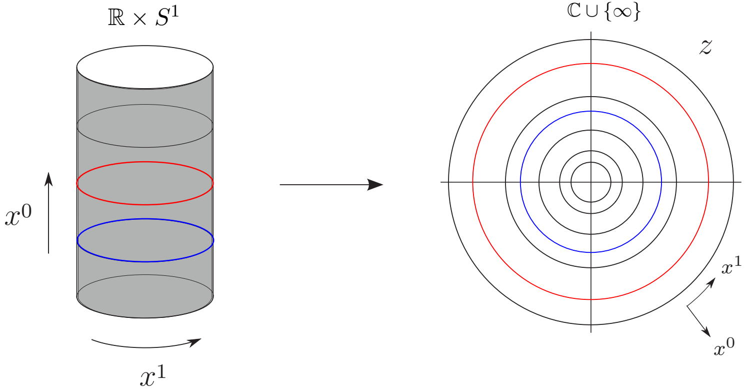
Consider a quantum field theory on two dimensional (flat) euclidian space with space and time coordinate x^0 and x^1.

- x^0: Time coordinate ranging form *infinite past* $(x^0 = -\infty)$ to *infinite future* $(x^0 = \infty)$
- x^1: Space coordinate, compactified by $x^1 \equiv x^1 + 2\pi$ and hence $x^1 \in [0, 2\pi)$
- This space is homeomorphic to an infinite cylinder $\mathbb{R} \times S^1$, which is also the world sheet of a closed string in euclidian space.

Map onto the Riemann sphere $\mathbb{C} \cup \{\infty\}$

We define the following conformal map on the cylinder:

$$\mathbb{R} \times S^1 \ni (x^0, x^1) \mapsto \exp(x^0 + ix^1) = z \in \mathbb{C} \cup \{\infty\}$$



Properties of Radial Quantization

- Infinite past and future: $x^0 = \mp\infty \mapsto z = 0, \infty$
- Equal time slices become circles of constant radius
- Time translations: $x^0 \to x^0 + a$ are dilations $z \to e^a z$
- Generator of dilations: Hamiltonian of the system
- Circles of constant radius: Hilbert space of the system

Primary Fields

Let $z \mapsto f(z)$ be a conformal transformation. If a field $\phi(z, \overline{z})$ transforms like

$$\phi(z,\overline{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi(f(z), \overline{f}(\overline{z}))$$

it is called a primary field of conformal weight $(h, \overline{h})$.
If this holds only for global conformal transformations it is called quasi-primary
For a small conformal transformation $w(z) = z + \epsilon(z)$ the corresponding infinitesimal transformation is

$$\delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z}) = \left[h(\partial_z\epsilon(z)) + \epsilon(z)\partial_z\right]\phi(z,\overline{z}) + \text{ anti-holom.}$$

Symmetries, conserved currents and charges

By Noether's Theorem, we have a **conserved current** associated to any continuous symmetry

$$j_\mu = T_{\mu\nu}\epsilon^\nu$$

The associated **conserved charge** is given by integration over the space coordiante(s).

$$Q = \int \mathrm{d}x^1 j_0$$

at $x^0 \equiv$ const

Infinitesimal symmetry variation

In radial quantization this means:

- $x^0 \equiv$ const $\longrightarrow |z|$=const
- $\int \mathrm{d}x^1 \longrightarrow \oint \mathrm{d}z$
- $T_{\mu\nu}\epsilon^\nu \rightarrow T(z)\epsilon(z) + \overline{T}(\overline{z})\overline{\epsilon}(\overline{z})$

$$Q_{\epsilon,\overline{\epsilon}} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \left(\mathrm{d}z T(z)\epsilon(z) + \mathrm{d}\overline{z}\overline{T}(\overline{z})\overline{\epsilon}(\overline{z})\right)$$

In QFT, the variation of a field is given by the **equal time commutator** of the conserved charge with the field.
Hence, for the infinitesimal symmetry variation of a field $\phi(z,\overline{z})$, we have:

$$\begin{aligned}\delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z}) &= [Q_{\epsilon,\overline{\epsilon}}, \phi(w,w)] \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \left(\mathrm{d}z\,[T(z)\epsilon(z), \phi(w,\overline{w})] + \mathrm{d}\overline{z}\,\left[\overline{T}(\overline{z})\overline{\epsilon}(\overline{z}), \phi(w,\overline{w})\right]\right)\end{aligned}$$

Radial Ordering

In a QFT, operator products need to be time ordered which, in radial quantization, corresponds to radial ordering and is defined as:

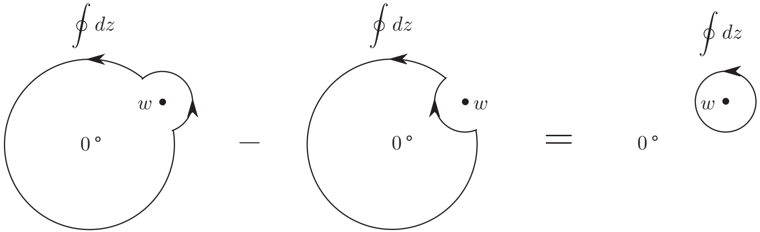
Radial Ordering Operator

For the product of two Operators A and B we define:

$$\mathcal{R}(A(z)B(w)) := \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases}$$

Deformation of the Integration Contour

$$\oint_{\mathcal{C}} \mathrm{d}z\, [A(z), B(w)] = \oint_{|z|>|w|} \mathrm{d}z A(z)B(w) - \oint_{|z|<|w|} \mathrm{d}z B(w)A(z)$$
$$= \oint_{\mathcal{C}(w)} \mathrm{d}z \mathcal{R}(A(z)B(w))$$



$$\begin{aligned}\delta_\epsilon \phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} \mathrm{d}z \epsilon(z) \mathcal{R}\left(T(z), \phi(w,\overline{w})\right) \\ &= h(\partial_w \epsilon(w))\phi(w,\overline{w}) + \epsilon(w)(\partial_w \phi(w,\overline{w})\end{aligned}$$

By Cauchy's formula, $\frac{1}{2\pi i}\oint_{\mathcal{C}(w)} \mathrm{d}z \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w)$, we have the following identities for the r.h.s terms:

$$\begin{aligned} h(\partial_w \epsilon(w))\phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} \mathrm{d}z \frac{h\epsilon(z)\phi(w,\overline{w})}{(z-w)^2} \\ \epsilon(w)\partial_w \phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{\mathcal{C}(w)} \mathrm{d}z \frac{h\epsilon(z)\partial_w\phi(w,\overline{w})}{z-w}\end{aligned}$$

$$\frac{1}{2\pi i}\oint_{\mathcal{C}(w)} \mathrm{d}z\epsilon(z)\mathcal{R}\left(T(z),\phi(w,\overline{w})\right)$$
$$=\frac{1}{2\pi i}\oint_{\mathcal{C}(w)} \mathrm{d}z\epsilon(z)\left(\frac{h}{(z-w)^2}\phi(w,\overline{w})+\frac{1}{z-w}\partial_w\phi(w,\overline{w})\right)$$

which leads to the following expression for our radially ordered product

OPE

$$\mathcal{R}\left(T(z),\phi(w,\overline{w})\right)\sim\frac{h}{(z-w)^2}\phi(w,\overline{w})+\frac{1}{z-w}\partial_w\phi(w,\overline{w})$$

where $\sim$ denotes the expansion up to in $\mathcal{C}(w)$ non-singular terms.

OPE: General Formulation

$$\mathcal{R}\left(A(x)B(w)\right) \sim \sum_i C_i(z-w)O_i(w)$$

where the O_i's are a complete set of local operators and the C_i's (singular) numerical coefficients.

OPE and Commutators

OPE of the the energy-momentum tensor with itself

Let $|z| > |w|$ and c denote the central charge

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)$$

This OPE can be used to obtain the commutator relations of the Virasoro-Algebra.

Laurent Expansion of $T(z)$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad \text{where} \quad L_n = \frac{1}{2\pi i}\oint \mathrm{d}z z^{n+1} T(z)$$

OPE and Commutators

Consider a particular conformal transformation $\epsilon(z) = -\epsilon_n z^{n+1}$ and express the conserved charge as

$$Q_n = \frac{1}{2\pi i} \oint \mathrm{d}z T(z)(-\epsilon_n z^{n+1}) = -\epsilon_n \sum_{m\in\mathbb{Z}} \oint \frac{\mathrm{d}z}{2\pi i} L_m z^{n-m-1}$$
$$= -\epsilon_n \sum_{m\in\mathbb{Z}} L_m \delta_{nm} 2\pi i = -\epsilon_n L_n$$

Hence, we have that $\delta_{\epsilon_n}\phi = -[Q_n, \phi] = -\epsilon[L_n, \phi]$ which means that the L_n are generators of conformal transformations on the Hilbertspace and can be identified with the generators l_n of the Witt-algebra

OPE and Commutators

$$
\begin{aligned}
[L_n, L_m] &= \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} z^{n+1} w^{m+1} [T(z), T(w)] \\
&= \oint_{\mathcal{C}(w)} \frac{\mathrm{d}z}{2\pi i} \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi i} z^{n+1} w^{m+1} \mathcal{R}(T(z)T(w)) \\
&= \oint_{\mathcal{C}(w)} \frac{\mathrm{d}z}{2\pi i} \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi i} z^{n+1} w^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) \right. \\
&\left. + \frac{1}{z-w} \partial_w T(w) + \cdots \right) \\
&= \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi i} \left(\frac{c/2}{3!} \partial^3 w^{n+1} + 2(\partial w^{n+1}) T(w) + w^{n+1} \partial T(w) \right) \\
&= (n-m) L_{n+m} + \frac{c}{12}(n^3 - n) \delta_{n+m,0}
\end{aligned}
$$

In- and Out-States

In-State

$$|A_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} A(z,\bar{z})|0\rangle$$

Adjoint

$$A(z,\bar{z})^\dagger = A\left(\frac{1}{z},\frac{1}{\bar{z}}\right)\frac{1}{\bar{z}^{2h}}\frac{1}{z^{2\bar{h}}}$$

Out-State

$$\langle A_{\text{out}}| = \lim_{z,\bar{z}\to 0}\langle 0|\widetilde{A}(z,\bar{z}) = \lim_{z,\bar{z}\to\infty}\langle 0|A(z,\bar{z})z^{2h}\bar{z}^{2\bar{h}}$$

In particular, it follows that $L_m^\dagger = L_{-m}$

Analogy to Angular Momentum Algebra $SU(2)$

- Generators: $J^{\pm}, J_z$
- States: Maximal set of commuting operators: J^2 and J_z
  - Casimir operator J^2-eigenvalue denotes representations (j)
  - $(2j+1)$-dimensional represenation space $V^{(j)}$
  - Eigenstates labeled by J_z-eigenvalues: $|j, m\rangle$
  - $J^{\pm}$ transforms between states in $V^{(j)}$
- Highest weight state(with maximal m) is annihilated by J^+:

$$\boxed{J^+|j, m_{\text{max}}\rangle = 0}$$

- All other states obtained by repeated action of J^- on $|j, m_{\text{max}}\rangle$
  - Only finitely many states: $(J^-)^{2j+1}|j, m_{\text{max}}\rangle = 0$

Application to Virasoro-Algebra

We proceed similarly to the $SU(2)$ case:

- Generators: L_n, c
- Unitarity condition: $L_n^\dagger = L_{-n}$
- States: Maximal set of commuting operators: c, L_0
  - Representations are labeled by central charge c
  - Each state inside a representation is denoted by $(h, \overline{h})$, the eigenvalue of L_0
  - $\Rightarrow |c, h, \overline{h}\rangle$ or for fixed central charge c just $|h, \overline{h}\rangle$

Action of L_n on states

Let $|\psi\rangle$ be a state with $L_0|\psi\rangle = h\psi$. The commutator with L_0 yields

$$[L_0, L_n] = -nL_n \quad \Leftrightarrow \quad L_0 L_n = -nL_n + L_n L_0$$

It follows:

$$L_0 L_n|\psi\rangle = (L_n L_0 - nL_n)|\psi\rangle = (h-n)L_n|\psi\rangle$$

We see that $L_{-n}|c,h\rangle$ has eigenvalue $(h+n)$ under L_0.
As for the case of J^-, other states can be obtained by successive application of L_{-n} for $n > 0$

Highest Weight Representation(HWR)

A HWR is a representation containing a state with smallest eigenvalue h of L_0 which is called a Highest Weight State.

- It is reasonable such an representation exists since the Hamiltonian $L_0 + \overline{L}_0$ is usually bounded
- Counterexample: Adjoint representation

Heighest Weight State

Due to the minimality requirement and the previously computed action of L_n, all $L_{n>0}$ must annihilate the highest weight state:

$$L_0|h, \overline{h}\rangle = h|h, \overline{h}\rangle \qquad L_{n>0}|h, \overline{h}\rangle = 0$$

Descendant States

As shown, the generators with negative n, $L_{n<0}$ can be used to generate other states in the given representation, by increasing the eigenvalue by n.

Descendant states

are states of the form

$$L_{-n_1} \cdots L_{-n_k} |c, h\rangle \quad n_i > 0$$

where $N = \sum_{i=1}^{k} n_i$ is called the Level of the state.

The set of all descendant states is called a Verma module $V_{c,h}$

Primary field - HW-state correspondence

Let $\phi(z,\overline{z})$ be a primary field of conformal weight $(h,\overline{h})$.
We define the state $|h,\overline{h}\rangle = \phi(0,0)|0\rangle$ generated by the field acting on the vacuum state and claim it is a HWS.

$$\begin{aligned}[L_n,\phi(w,\overline{w})] &= \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z)\phi(w,\overline{w}) \\ &= \oint \frac{\mathrm{d}z}{2\pi i}\left(\frac{hz^{n+1}}{(z-w)^2}\phi(w,\overline{w}) + \frac{z^{n+1}}{z-w}\partial_w\phi(w,\overline{w}) + \cdots\right) \\ &= h(n+1)w^n\phi(w,\overline{w}) + w^{n+1}\partial_w\phi(w,\overline{w}) = 0 \quad \text{for} \quad w=0\end{aligned}$$

It follows the annihilation condition for $w=0$ and $n>0$:

$$L_n|h,\overline{h}\rangle = L_n\phi(0,0)|0\rangle = \phi(0,0)L_n|0\rangle + [L_n,\phi(0,0)]|0\rangle = 0$$

Primary field - HW-state correspondence

Now, for $n = 0$ the commutator yields $[L_0, \phi(0,0)] = h\phi(0,0)$ which means that

$$\begin{aligned} L_0|h, \overline{h}\rangle &= L_0\phi(0,0)|0\rangle = \phi(0,0)L_0|0\rangle + [L_0, \phi(0,0)]|0\rangle \\ &= h\phi(0,0)|0\rangle = h|h, \overline{h}\rangle \end{aligned}$$

This proofs that $|h, \overline{h}\rangle = \phi(0,0)|0\rangle$ is indeed a highest weight state.

Descendant states and descendant fields

Start with a descendant state $L_{-k}|h,h'\rangle$

$$\begin{aligned} L_{-k}|h,h'\rangle &= L_{-k}\phi(0,0)|0\rangle \\ &= \overbrace{\oint \frac{\mathrm{d}z}{2\pi i} z^{-k+1} T(z)}^{L_{-k}} \phi(0,0)|0\rangle \\ &= \phi^{(-k)}(0,0)|0\rangle \end{aligned}$$

This motivates the definition of a corresponding descendant field as

$$\phi^{(-k)}(w,\overline{w}) = \oint \frac{\mathrm{d}z}{2\pi i} \frac{T(z)\phi(w,\overline{w})}{(z-w)^{k-1}}$$

Since L_0 commutes with L_n we can formally assign a conformal weight $(h+k,\overline{h})$ to this field which is given by the L_0 eigenvalue.

Example: Descendant Field of the Identity

Consider the identity field: $\mathbb{1}$

$$(L_{-2}\mathbb{1})(w) = \oint \frac{\mathrm{d}z}{2\pi i} \frac{1}{z-w} T(z)\mathbb{1} = T(w)$$

We see that $\mathbb{1}^{(-2)}(w) = (L_{-2}\mathbb{1})(w) = T(w)$ is a level 2 descendant of the identity operator.

Descendant fields as derivatives

Write the complete(including non-singular terms) OPE of T(z) with a primary field as

$$T(z)\phi(w,\overline{w}) = \sum_{n\geq 0}(z-w)^{n-2}\overbrace{L_{-n}\phi(w,\overline{w})}^{\phi^{(-n)}}$$
$$= \frac{1}{(z-w)^2}L_0\phi + \frac{1}{z-w}L_{-1}\phi + L_{-2}\phi + (z-w)L_{-3}\phi + \cdots$$

Compare to previously computed OPE:

$$T(z)\phi(w,\overline{w}) = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) + \dots$$

We see that $\phi^{(0)} = L_0\phi = h\phi(w,\overline{w})$ and $\phi^{(-1)} = L_{-1}\phi = \partial_w\phi(w,\overline{w})$

Conformal Family

The set comprising a primary field ϕ and all of its descendants is called a conformal family, and is denoted by:

$$[\phi] = \{\phi, (L_{-n}\phi), \ldots, (L_{-k_1} \cdots L_{-k_N}\phi); n > 0, k_i > 0\}$$

level	dimension	field
0	h	ϕ
1	h+1	$L_{-1}\phi$
2	h+2	$L_{-2}\phi, L^2_{-1}\phi$
3	h+3	$L_{-3}\phi, L_{-1}L_{-2}\phi, \phi, L^3_{-1}\phi$
⋮	⋮	⋮
N	h+N	P(N) fields

where $P(N)$ denotes the number of partitions of N into positive integers.

BACKUP

Application: Correlation functions

Correlation functions with descedants can be reduced to correlation function of their primary fields:
Let X be $\phi_1(w_1)\cdots\phi_n(w_n)$ a product of primary fields with conformal dimension $(h_i, 0)$.

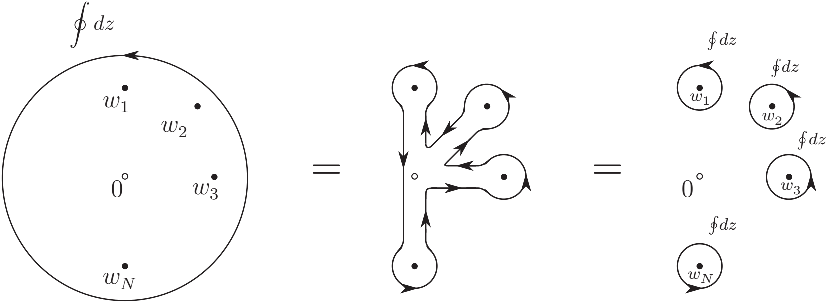
$$\langle \phi^{(-k)}(w)X\rangle := \langle \phi^{(-k)}(w)\phi_1(w_1)\dots\phi_n(w_n)\rangle$$

$$= \oint_{\mathcal{C}(w)} \frac{\mathrm{d}z}{2\pi i}(z-w)^{-n+1}\langle T(z)\phi(w)X\rangle$$

$$= -\sum_i \oint_{\mathcal{C}(w_i)} \frac{\mathrm{d}z}{2\pi i}(z-w)^{-n+1}\langle \phi(w)\phi_1(w_1)\cdots T(z)\phi(i)\cdots\phi_n(w_n)\rangle$$

$$= -\sum_i \oint_{\mathcal{C}(w_i)} \left(\frac{h_i(z-w)^{-n+1}}{(z-w_i)^2} + \frac{(z-w)^{-n+1}}{z-w_i}\partial_{w_i}\right)\langle\phi(w)X\rangle$$

$$= -\sum_i \left(h_i(1-n)(w_i-w)^{-n} + (w_i-w)^{-n+1}\partial_{w_i}\right)\langle\phi(w)X\rangle$$


$\oint dz$
w_1
w_2
w_3
0
w_N
$=$
$=$
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w_1
w_2
0
w_3
w_N

Contour Deformation

$$
\begin{aligned}
&= -\sum_i \oint_{\mathcal{C}(w_i)} \frac{\mathrm{d}z}{2\pi i}(z-w)^{-n+1}\langle\phi(w)\phi_1(w_1)\cdots T(z)\phi(i)\cdots\phi_n(w_n)\rangle \\
&= -\sum_i \oint_{\mathcal{C}(w_i)} \left(\frac{h_i(z-w)^{-n+1}}{(z-w_i)^2} + \frac{(z-w)^{-n+1}}{z-w_i}\partial_{w_i}\right)\langle\phi(w)X\rangle \\
&= -\sum_i \left(h_i(1-n)(w_i-w)^{-n} + (w_i-w)^{-n+1}\partial_{w_i}\right)\langle\phi(w)X\rangle \\
&=: \mathcal{L}_{-n}\langle\phi(w)X\rangle
\end{aligned}
$$

Hence, correlation functions of descendant fields are given by differential operators acting in their associated primary fields.

Mode Expansion of field

Expand an arbitrary holomorphic primary field $\phi(z)$ with weight (h,0):

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \quad \text{with} \quad \phi_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{h+n-1}\phi(z)$$

From regularity of $\phi(z)|0\rangle$ at $z=0$ follows that $\phi_n|0\rangle = 0$ for $n \geq -h+1$ and
$|h\rangle = \phi_{-h}|0\rangle$

Mode-Generator Commutator

$$
\begin{aligned}
[L_n, \phi_m] &= \oint \frac{\mathrm{d}w}{2\pi i} w^{h+m-1} (h(n+1) w^n \phi(w) + w^{n+1} \partial\phi(w)) \\
&= \oint \frac{\mathrm{d}w}{2\pi i} w^{h+m+n-1} (h(n+1) - (h+m+n)) \phi(w) \\
&= (n(h-1) - m) \phi_{m+n} \\
\Rightarrow [L_0, \phi_m] &= -m \phi_m
\end{aligned}
$$