REPORT

Low energy effective actions from String Theory

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Abstract

We consider the general non-linear sigma model in $(1+1)$ dimensions as a starting point to studying the relativistic string. The Weyl invariance of the classical action is violated at the quantum level, leading to an anomaly. We explicitly calculate the Weyl anomaly coefficient to first order in perturbation theory, and restore Weyl invariance by imposing a suitable condition. The Weyl anomaly condition is shown to follow from an emergent action principle that fixes the target space to obey Einstein's equations in vacuum. A generalisation of the Polyakov action is briefly discussed.

Contents

1 Introduction

String theory has long been considered as a possible candidate for a theory of everything, aiming to unify the Standard Model of particle physics with a quantum theory of gravity. It is arguably the most popular of all candidate theories and has been the focus of intensive research for the past few decades. The aim of this report is to give an introductory treatment of how gravity emerges in the context of the simplest string theory model, bosonic string theory.

We will start by reviewing the Polyakov action of the classical, relativistic string. Briefly discarding the string interpretation, this is simply the action of a non-linear, two-dimensional conformal field theory also known as a sigma model. A discussion of the symmetries of the classical sigma model naturally raises the question whether these symmetries will be preserved in the quantum version of the theory. We give arguments to show that at least one of these symmetries, namely Weyl invariance, will be violated in the quantum theory, which will lead to the so called Weyl anomaly. The next section is dedicated to explicitly calculating the Weyl anomaly to one-loop order in perturbation theory. The end result of this report will be to show that the Weyl invariance of the quantum theory can be restored by imposing suitable restrictions on the physical target space in which the string dynamics take place. The restrictions will lead to an action and equations of motion for the target space, thus recovering the original interpretation of the Polyakov action as describing a string living on a dynamic, gravitational background.

The majority of this report is based on the 1988 TASI Summer School lecture notes "*Sigma Models and String Theory*" by C. Callan and L. Thorlacius [1].

2 The Classical String

2.1 The Polyakov Action

A classical string is simply a 1-dimensional object propagating in some D-dimensional spacetime. One possible action to describe the dynamics of such a string is the Polyakov action, given by

$$S_P = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \tag{2.1}$$

This describes the invariant area of the $(1+1)$-dimensional string worldsheet with metric γ_{ab} $(a, b = 0, 1)$ swept out by some embedding $X^\mu(\sigma, \tau)$ $(\mu = 0, 1, \ldots, D)$ of the string into a target space that has a metric $G_{\mu\nu}(X)$. The parameter α' has units of $[\text{mass}]^{-2}$ or alternatively $[\text{length}]^2$, and the energy or length scale it sets depends on the physics the string theory is designed to describe.

There is another action, classically equivalent to S_P, which shows more clearly the geometric origins of the string action as describing an area in spacetime. This so called Nambu-Goto action[2], however, involves the square root of derivatives and is therefore hideously non-linear. On the contrary, the Polyakov action is quadratic in derivatives of the X's and will thus be easier to quantize.

The interpretation of the X's and the metric $G_{\mu\nu}$ as describing the embedding of the string in a physical spacetime is not a priori inserted into the string theory model. In what follows, we will instead consider a two-dimensional quantum field theory, consisting of D scalar variables X^μ living on some crumpled two-dimensional surface with a metric γ_{ab}. The action for the X^μ fields constains kinetic and potential terms which are described abstractly by the $G_{\mu\nu}$ term. For particular choices of $G_{\mu\nu}$ we recover a class of actions which are familiar from other contexts; for example, if one chooses $G_{\mu\nu}$ to be the metric on the D-dimensional sphere then this becomes the $SO(D)$ non-linear sigma model studied in the context of spontaneous symmetry breaking. The name non-linear sigma model has carried over to describe actions with a non-specific $G_{\mu\nu}$. We will see later that we must make restrictions on the properties of $G_{\mu\nu}$ in order for the two-dimensional QFT (2.1) to describe a satisfactory string theory, particularly when we try to quantize the theory.

2.2 Symmetries of the Polyakov Action

The Polyakov action has two important symmetries. First of all, it is reparametrisation invariant, which means invariance under any type of general coordinate transformation on the worldsheet. If we denote the coordinates (σ, τ) as σ^a, $a \in \{0, 1\}$, we have under a transformation

$$\sigma^a \to \tilde{\sigma}^a(\sigma) \tag{2.2}$$

that the kinetic term in the action transform as (we briefly omit the index μ for clarity)

$$\begin{aligned}\tilde{\gamma}^{ab}\tilde{\partial}_a X \tilde{\partial}_b X &= \gamma^{cd}\frac{\partial\tilde{\sigma}^a}{\partial\sigma^c}\frac{\partial\tilde{\sigma}^b}{\partial\sigma^d}\frac{\partial}{\partial\tilde{\sigma}^a}X\frac{\partial}{\partial\tilde{\sigma}^b}X \\ &= \gamma^{cd}\partial_c X \partial_d X\end{aligned}$$

whereas the metric determinant $\gamma = \det(\gamma_{ab})$ and the integration measure $\mathrm{d}^2\sigma$ transform with the Jacobian determinant $J = \det\left(\frac{\partial\tilde{\sigma}^a}{\partial\sigma^b}\right)$:

$$\begin{aligned}\mathrm{d}^2\tilde{\sigma} &= J\mathrm{d}^2\sigma \\ \tilde{\gamma} &= J^{-2}\gamma\end{aligned}$$

Putting everything together, the transformation of the measure precisely cancels that of the metric determinant, and the action (2.1) is shown to be invariant under the transformation (2.2). From Noether's theorem one can then derive that there is an associated conserved current; this is the worldsheet energy momentum tensor

$$T_{ab} = \frac{4\pi}{\sqrt{\gamma}}\frac{\delta S_P}{\delta\gamma^{ab}} \tag{2.3}$$

which is covariantly conserved:

$$\nabla^a T_{ab} = 0 \tag{2.4}$$

Note that the Euler-Lagrange equation of motion for the worldsheet metric γ_{ab} is simply $T_{ab} = 0$. This is seemingly a stronger requirement than the conservation equation (2.4) but whereas the equation of motion can be violated for off-shell γ_{ab}, the equation (2.4) will always hold in a reparametrisation invariant theory.

The Polyakov action is also invariant to a second class of transformations, the Weyl transformations. A Weyl transformation is a local rescaling of the metric, i.e. a transformation which chances the distances locally but preserves angles. Under a Weyl transformation the worldsheet metric transforms as

$$\tilde{\gamma_{ab}} = e^{\phi(\sigma)}\gamma_{ab} \tag{2.5}$$

This means that the inverse metric γ^{ab} transforms as $e^{-\phi(\sigma)}\gamma^{ab}$ whereas the metric determinant will transform as

$$\begin{aligned}\tilde{\gamma} &= \det(\tilde{\gamma_{ab}}) && (2.6)\\ &= (e^{\phi(\sigma)})^2 \det(\gamma_{ab}) && (2.7)\\ &= e^{2\phi(\sigma)}\gamma && (2.8)\end{aligned}$$

Since a Weyl transformation does not affect the fields X^μ this shows that S_P has Weyl invariance. Note that Weyl invariance depends crucially on the fact that γ_{ab} is two-dimensional; in an arbitrary dimension d the metric determinant will transform with a power of d of the Weyl scaling factor and will not cancel out the contribution from the metric itself.

Weyl transformation immediately implies that the trace of the energy momentum tensor is identically zero. Since the action should not change under a variation with respect to the the scale factor ϕ, we have

$$\begin{aligned}0 = \frac{\delta S_P}{\delta\phi} &= \frac{\delta S_P}{\delta\gamma^{ab}}\frac{\delta\gamma^{ab}}{\delta\phi} && (2.9)\\ &= -\phi(\sigma)\frac{\sqrt{\gamma}}{4\pi}T_{ab}\gamma^{ab} \sim T^a_{\ a} && (2.10)\end{aligned}$$

2.3 The Weyl Anomaly

A question that naturally occurs when trying to quantize a field theory is whether the symmetries of the classical field theory are also exact symmetries of the quantized field theory. In general, the answer to this is “not always” and the resulting loss of symmetry is called an Anomaly. Anomalies occur in a wide variety of contexts, with one of the most well known ones being the chiral or axial anomaly in Quantum Electrodynamics, which explains for example the observed decay rate of the neutral pion (see [3] for a detailed discussion).

In the case of the Polyakov action (2.1), one can show [1] that the Weyl symmetry is anomalous at the quantum level. The trace of the energy momentum tensor, once promoted to an operator in the quantum field theory, will acquire a non-zero expectation value. However, it turns out that the Weyl invariance of the quantized action can be salvaged by imposing suitable constraints. In the next section we will compute the one-loop contribution to the Weyl anomaly and give an explicit form for these constraints.

3 Quantisation

3.1 Covariant Background Field Expansion

We want to perform a perturbative expansion of the action S_P. The metric $G_{\mu\nu}$ transforms covariantly under spacetime general coordinate transformations and we would like to maintain this feature in the expansion. In this way, we can calculate diagrams using covariant vertices and propagators so that the Weyl anomaly coefficient can be expressed in a coordinate invariant way. The way to do this is to separate the fields X^μ in a background part and a quantum part

$$X^\mu(\sigma) = X_0^\mu(\sigma) + \pi^\mu(\sigma) \tag{3.1}$$

where the background field X_0^μ is a stationary point of the action, i.e. a solution to the classical equations of motion

$$\frac{\delta S}{\delta X^\mu} = 0 \tag{3.2}$$

This way, the quantum part represents a "small" quantum fluctuation around the classical solution and we can perform a perturbative expansion in powers of π^μ.

The immediate problem is that the quantum field π^μ is defined as a coordinate difference $X^\mu - X_0^\mu$ and therefore does not transform as a vector under general coordinate transformations. The solution to this is to replace π^μ with the tangent vector η^μ to the spacetime geodesic which connects the points X_0^μ and $X_0^\mu + \pi^\mu$. The η^μ define a system of coordinates called Riemann Normal Coordinates which has several nice properties.

For a small enough neighbourhood around X_0^μ there is a unique geodesic $\lambda^\mu(t)$ that connects the points X_0^μ and $X_0^\mu + \pi^\mu$, where t is a real parameter chosen such that ($\dot{} = \frac{\mathrm{d}}{\mathrm{dt}}$)

$$\begin{aligned}
\lambda^\mu(0) &= X_0^\mu \\
\lambda^\mu(1) &= X_0^\mu + \pi^\mu \\
\dot{\lambda}^\mu(0) &= \eta^\mu
\end{aligned}$$

The geodesic equation for $\lambda^\mu(t)$ is

$$\ddot{\lambda}^\mu(t) + \Gamma^\mu_{\nu\sigma}\dot{\lambda}^\nu(t)\dot{\lambda}^\sigma(t) = 0 \tag{3.3}$$

At the point X_0^μ $(t=0)$ this becomes

$$\Gamma^\mu_{\nu\sigma}\eta^\nu\eta^\sigma = 0 \tag{3.4}$$

Since this has to hold for all choices of η^μ, it tells us that $\Gamma^\mu_{\nu\sigma} = 0$ at the point X_0^μ. This is simply an expression of the statement that any pseudo-Riemannian manifold is locally flat, and as such we can always choose coordinates such that the Christoffel symbols and by extension the curvature are zero at a point. We now Taylor expand $\lambda^\mu(t)$ around $t=0$

$$\lambda^\mu(t) = X_0^\mu + \eta^\mu t - \frac{1}{2}\Gamma^\mu_{\nu\sigma}\eta^\nu\eta^\sigma + \ldots \tag{3.5}$$

At $t=1$ this defines a coordinate transformation from π^μ to the Riemann Normal Coordinates (RNC) η^μ, and we have

$$\pi^\mu = \eta^\mu - \frac{1}{2}\Gamma^\mu_{\nu\sigma}\eta^\nu\eta^\sigma + \ldots \tag{3.6}$$

The curvature tensor is defined as

$$R^\mu{}_{\nu\sigma\rho} = \partial_\sigma\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\sigma} + \Gamma^\lambda_{\rho\nu}\Gamma^\mu_{\lambda\sigma} - \Gamma^\lambda_{\sigma\nu}\Gamma^\mu_{\lambda\rho}$$

In RNC (where we denote the relevant quantities with a bar on top) this reduces to

$$\bar{R}^\mu{}_{\nu\sigma\rho} = \partial_\sigma\bar{\Gamma}^\mu_{\nu\rho} - \partial_\rho\bar{\Gamma}^\mu_{\nu\sigma} \tag{3.7}$$

and we can invert this expression to express the derivatives of the Christoffel symbols in terms of the curvature:

$$\begin{aligned}\bar{R}^\mu{}_{\nu\sigma\rho} + \bar{R}^\mu{}_{\rho\sigma\nu} &= \partial_\sigma\bar{\Gamma}^\mu_{\nu\rho} - \partial_\rho\bar{\Gamma}^\mu_{\nu\sigma} + \partial_\sigma\bar{\Gamma}^\mu_{\rho\nu} - \partial_\nu\bar{\Gamma}^\mu_{\rho\sigma} \\ &= 2\,\partial_\sigma\bar{\Gamma}^\mu_{\nu\rho} - \partial_\rho\bar{\Gamma}^\mu_{\nu\sigma} - \partial_\nu\bar{\Gamma}^\mu_{\rho\sigma}\end{aligned}$$

this can be simplified further by taking the derivative of (3.4) in the direction η

$$\begin{aligned}0 &= \eta^\rho\partial_\rho\bar{\Gamma}^\mu_{\nu\sigma}\eta^\nu\eta^\sigma && (3.8)\\ &= \partial_\rho\bar{\Gamma}^\mu_{\nu\sigma}\eta^\nu\eta^\sigma\eta^\rho && (3.9)\end{aligned}$$

Since $\eta^\nu\eta^\sigma\eta^\rho$ is a symmetric tensor, we can symmetrize the indices of Γ

$$\begin{aligned}0 &= \partial_{(\rho}\bar{\Gamma}^\mu_{\nu\sigma)} && (3.10)\\ &= \partial_\rho\bar{\Gamma}^\mu_{\nu\sigma} + \partial_\nu\bar{\Gamma}^\mu_{\rho\sigma} + \partial_\sigma\bar{\Gamma}^\mu_{\nu\rho} && (3.11)\end{aligned}$$

Plugging this into (3.8) we obtain

$$\partial_\sigma\bar{\Gamma}^\mu_{\nu\rho} = \frac{1}{3}\left(\bar{R}^\mu{}_{\nu\sigma\rho} + \bar{R}^\mu{}_{\rho\sigma\nu}\right) \tag{3.12}$$

Using the new coordinates η^μ, we can now perform an explicitly covariant Taylor expansion of an arbitrary tensor around the point X_0^μ. For a 2-covariant tensor,

$$\bar{T}_{\mu\nu}(X_0+\pi) = \bar{T}_{\mu\nu}(X_0) + \eta^\lambda \partial_\lambda \bar{T}_{\mu\nu}(X_0) + \frac{1}{2}\eta^\lambda\eta^\sigma\partial_\lambda\partial_\sigma \bar{T}_{\mu\nu}(X_0) + \mathcal{O}(\eta^3) \quad (3.13)$$

For the first order term we can replace the partial derivative with a covariant derivative and a term involving the Christoffel symbol, but in RNC the latter will vanish, leaving

$$\eta^\lambda\partial_\lambda\bar{T}_{\mu\nu} = \eta^\lambda\nabla_\lambda\bar{T}_{\mu\nu} \quad (3.14)$$

For the second order term, we have

$$\eta^\lambda\eta^\sigma\partial_\lambda\partial_\sigma\bar{T}_{\mu\nu} = \eta^\lambda\eta^\sigma\partial_\lambda\left(\nabla_\sigma\bar{T}_{\mu\nu} + \bar{\Gamma}^\rho_{\sigma\mu}\bar{T}_{\rho\nu} + \bar{\Gamma}^\rho_{\sigma\nu}\bar{T}_{\mu\rho}\right) \quad (3.15)$$

Again replacing ∂_λ with the covariant derivative and removing terms that involve the Christoffel symbol (but keeping the ones that involve its derivatives), this becomes

$$\eta^\lambda\eta^\sigma\partial_\lambda\partial_\sigma\bar{T}_{\mu\nu} = \eta^\lambda\eta^\sigma\left\{\nabla_\lambda\nabla_\sigma\bar{T}_{\mu\nu} + \left(\partial_\lambda\bar{\Gamma}^\rho_{\sigma\mu}\right)\bar{T}_{\rho\nu} + \left(\partial_\lambda\bar{\Gamma}^\rho_{\sigma\nu}\right)\bar{T}_{\mu\rho}\right\} \quad (3.16)$$

Consider the second term,

$$\eta^\lambda\eta^\sigma\left(\partial_\lambda\bar{\Gamma}^\rho_{\sigma\mu}\right)\bar{T}_{\rho\nu}(X_0) = \frac{1}{3}\eta^\lambda\eta^\sigma\left(\bar{R}^\rho{}_{\sigma\lambda\mu} + \bar{R}^\rho{}_{\mu\lambda\sigma}\right)\bar{T}_{\rho\nu}(X_0) \quad (3.17)$$

$$= \frac{1}{3}\eta^\lambda\eta^\sigma\left(\bar{R}^\rho{}_{\sigma\lambda\mu}\right)\bar{T}_{\rho\nu}(X_0) \quad (3.18)$$

where the second curvature tensor cancels because of the antisymmetry $R_{\rho\mu\lambda\sigma} = -R_{\rho\mu\sigma\lambda}$ and contracting with the symmetric tensor $\eta^\sigma\eta^\lambda$.

The last term is analogous and finally the covariant Taylor expansion reads

$$\bar{T}(X_0+\pi) = \bar{T}_{\mu\nu}(X_0) + \eta^\lambda\nabla_\lambda\bar{T}_{\mu\nu}(X_0) \quad (3.19)$$

$$+\frac{1}{2}\eta^\lambda\eta^\sigma\left(\nabla_\lambda\nabla_\sigma\bar{T}_{\mu\nu} - \frac{1}{3}\bar{R}^\rho{}_{\lambda\mu\sigma}\bar{T}_{\rho\nu} - \frac{1}{3}\bar{R}^\rho{}_{\lambda\nu\sigma}\bar{T}_{\mu\rho}\right)(X_0) + \mathcal{O}(\eta^3) \quad (3.20)$$

The expansion only involves covariant derivatives and spacetime tensors so it holds in any coordinate system, which means we can drop the bars from the notation. We can now apply this expansion to the ingredients of the action S_P. The target space metric $G_{\mu\nu}$ has a simple expansion since it is assumed to be symmetric and covariantly constant, i.e. $\nabla G = 0$. Using (3.20) we get

$$G_{\mu\nu}(X_0+\pi) = G_{\mu\nu}(X_0) - \frac{1}{6}\eta^\lambda\eta^\sigma\left(R^\rho{}_{\lambda\mu\sigma}G_{\rho\nu} + R^\rho_{\lambda\nu\sigma}G_{\mu\rho}\right) + \mathcal{O}(\eta^3) \quad (3.21)$$

Since the left hand side is symmetric in μ and ν we can symmetrize the right hand side, and we use $G_{\mu\nu}$ to bring down the spacetime index of the curvature to get

$$G_{\mu\nu}(X_0+\pi) = G_{\mu\nu}(X_0) + \frac{1}{3}\eta^\lambda \eta^\sigma R_{\mu\lambda\nu\sigma} + \mathcal{O}(\eta^3) \tag{3.22}$$

The expansion of $\partial_a X^\mu$ is a bit more involved and ends up taking the form (see [5] for a full derivation)

$$\partial_a(X_0^\mu + \pi^\mu) = \partial_a X_0^\mu + \nabla_a \eta^\mu + \frac{1}{3} R^\mu_{\lambda\sigma\nu}(X_0)\partial_a X_0^\nu \eta^\lambda \eta^\sigma + \mathcal{O}(\eta^3) \tag{3.23}$$

Putting everything together, we have

$$\begin{aligned}
&G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \\
&\quad = G_{\mu\nu}(X_0+\pi)\partial_a(X_0+\pi)^\mu \partial_b(X_0+\pi)^\nu \\
&\quad = \left(G_{\mu\nu}(X_0) + \frac{1}{3}R_{\mu\lambda\nu\sigma}\eta^\lambda\eta^\sigma\right) \\
&\qquad \times \left(\partial_a X_0^\mu + \nabla_a \eta^\mu + \frac{1}{3}R^\mu_{\lambda\sigma\nu}(X_0)\partial_a X_0^\nu \eta^\lambda \eta^\sigma\right) \\
&\qquad \times \left(\partial_b X_0^\nu + \nabla_b \eta^\nu + \frac{1}{3}R^\nu_{\lambda\sigma\mu}(X_0)\partial_b X_0^\mu \eta^\lambda \eta^\sigma\right) \\
&\qquad + \dots \\
&\quad = G_{\mu\nu}(X_0)\left(\partial_a X_0^\mu \partial_b X_0^\nu + \partial_a X_0^\mu \nabla_b \eta^\nu + \partial_b X_0^\nu \nabla_a \eta^\nu + \nabla_a \eta^\mu \nabla_b \eta^\nu\right) \\
&\qquad + R_{\mu\lambda\sigma\nu}\partial_a X_0^\mu \partial_b X_0^\nu \eta^\lambda \eta^\sigma + \dots
\end{aligned}$$

In the action S_P, the terms linear in X_0 are of no interest. Indeed, we defined X_0 as being the solution to the classical equations of motion (3.2), so by definition S_P cannot depend linearly on X_0. Up to order η^2, the expansion of S_P is thus

$$S_P[X_0+\pi] = S_P[X_0] + \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{\gamma}\gamma^{ab}\left(G_{\mu\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu + R_{\mu\lambda\sigma\nu}\partial_a X_0^\mu \partial_b X_0^\nu \eta^\lambda\eta^\sigma\right) \tag{3.24}$$

One more problem needs to be addressed before we can quantize the action. The term involving $\nabla\eta\nabla\eta$ is quadratic in derivatives of the quantum field and represents the kinetic term of the theory. However, it involves the target space metric $G_{\mu\nu}$ which depends on X_0 and it would be difficult to extract a propagator from it. Therefore, we carry out one last coordinate transformation by introducing a vielbein $e^i_\mu(X_0)$, which is a linear operator that maps the vectors η^μ to a local Lorentz frame η^i ($i = 0, 1, \dots, D$),

$$\eta^i = e^i_\mu(X_0)\eta^\mu \tag{3.25}$$

The vielbein satisfies

$$e^i_\mu(X_0)e^j_\nu(X_0)\delta_{ij} = G_{\mu\nu}(X_0) \tag{3.26}$$

where δ_{ij} is a flat metric on D-dimensional space. That this is possible is once again a consequence of the local flatness of (pseudo-)Riemannian manifolds. In the η^i coordinate system, the kinetic term becomes

$$G_{\mu\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu = (\nabla_a\eta)^i(\nabla_b\eta)_i \tag{3.27}$$

which is now explicitly diagonal. The price we pay is to break gauge (diffeomorphism) invariance, since the local Lorentz frame defined by (3.26) is not invariant under a general coordinate transformation. However, we maintain explicit covariance in terms of the background X_0^μ fields, which is the point of the covariant background field expansion.

3.2 The Weyl Anomaly at one-loop

As mentioned in section 2.3, at the quantum level the energy-momentum tensor T_{ab} is promoted to an operator and will acquire an expectation value $\langle T_{ab}\rangle$. In the canonical quantisation framework [3], this reduces to evaluation an expression of the form

$$\langle 0 \mid \mathbb{T}\left\{T_{ab}R_{\mu\lambda\sigma\nu}\partial_c X_0^\mu \partial^c X_0^\nu \eta^\lambda\eta^\sigma\right\} \mid 0\rangle = \text{sum over Wick contractions} \tag{3.28}$$

where $\mathbb{T}$ denotes the time-ordering operator and we have introduced one copy of the energy momentum tensor T_{ab} and one copy of the first relevant term involving the background fields in the expansion of S_P. Let x, y and z denote worldsheet coordinates. The only nontrivial Wick contraction we need is then

$$\eta^i(x)\eta^j(y) = i\Delta^{ij}(x-y) \tag{3.29}$$

where Δ is the propagator of the η fields in position space. The standard propagator for a vector field η^i with diagonal kinetic term is

$$\Delta^{ij}(x) = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\delta^{ij}e^{ipx}}{p^2} = \int \frac{\mathrm{d}^2p}{(2\pi)^2}e^{ipx}\Delta^{ij}(p) \tag{3.30}$$

To find the expectation value of the trace of the energy momentum tensor, we switch to worldsheet light cone coordinates defined by

$$\sigma^\pm = \sigma^0 \pm \sigma^1$$
$$\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$$

In these coordinates, the conservation equation $\nabla^a \langle T_{ab} \rangle = 0$ becomes (the $b = 0$ component)

$$\partial_- \langle T_{++} \rangle + \partial_+ \langle T_{-+} \rangle = 0 \tag{3.31}$$

Since $T^a_{\ a} = -4T_{+-}$, we can calculate the expectation value of the trace of the energy momentum tensor by calculating the expectation value of the $(++)$ component and using equation (3.31). Going to momentum space, we get

$$q_- \langle T_{++} \rangle + q_+ \langle T_{-+} \rangle = 0 \tag{3.32}$$

The conservation equation (3.31) or alternatively (3.32) will hold even in the quantum theory if we insist on conservation of energy and momentum. Since $\langle T_{++} \rangle$ is in general non-zero this indeed implies that $\langle T_{+-} \rangle$ also has to be non-zero, and we see that in order to maintain conservation of energy and momentum we have to give up Weyl invariance at the quantum level (at least for now). Continuing with the calculation, to first order in S_P we have to calculate a diagram of the form shown in figure 3.1.

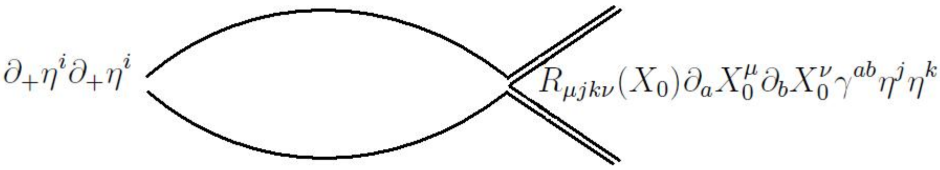

Figure 3.1: Feynman diagram corresponding to the first loop order in (3.28)

The factor $\partial_+\eta^i\partial_+\eta^i$ on the left comes from the insertion of T_{++}. The lines forming the loop represent the propagators (3.29) of the η^i. We insert T_{++} on the left hand side with a momentum q, and this momentum is carried away by the background fields (represented by double lines) X_0 on the right hand side. In the loop however, we can have an arbitrary momentum l running around which needs to be integrated over. The relevant contribution is, in momentum space (using the inverse Fourier transform of (3.30))

$$\int \mathrm{d}x\mathrm{d}y\, e^{iqx} \ :\ \partial_+\eta^i(x)\partial_+\eta^i(x) R_{\mu jk\nu}\partial_c X_0^\mu \partial^c X_0^\nu \eta^j(y)\eta^k(y)\ : \tag{3.33}$$

where everything betwen : : is contracted in all possible ways. There is only one way to contract the indices, which leads to

$$\int \mathrm{d}x\mathrm{d}y\, R_{\mu jk\nu}\partial_c X_0^\mu \partial^c X^\nu\, e^{iqx}\partial_+\Delta^{ij}(x-y)\partial_+\Delta^{ik}(x-y) \tag{3.34}$$

Using the expressions (3.30) for the η-propagators and assigning momenta l_1, l_2 to the two legs we get

$$\int \mathrm{d}x\mathrm{d}y \frac{\mathrm{d}l_1}{(2\pi)^2}\frac{\mathrm{d}l_2}{(2\pi)^2}\, R_{\mu jk\nu}\partial_c X_0^\mu \partial^c X^\nu\, \delta^{ij}\delta^{jk}\, e^{i(q+l_1+l_2)x} e^{i(l_1+l_2)y} \frac{(l_1)_+(l_2)_+}{l_1^2 l_2^2} \tag{3.35}$$

Integrating out the delta functions lets us set $l_1 + l_2 = -q$ and $l_1 = -l_2 = l$ and the final expression for diagram (3.1) is

$$\int \frac{\mathrm{d}^2 l}{2\pi} \frac{l_+(l_+ + q_+)}{l^2(l+q)^2} \left\{ \delta^{jk} R_{\mu jk\nu} \partial_a X_0^\mu \partial_b X_0^\nu \gamma^{ab} \right\} (q) \tag{3.36}$$

The part between curly brackets depends only on q and can be brought outside the integral. The integral (3.36) can be solved using dimensional regularisation. From [4] we get the following general formulae:

$$\int \mathrm{d}^N l \frac{l_\mu}{(l^2 + 2p \cdot l)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{-p_\mu}{p^{2A-N}} \tag{3.37}$$

$$\int \mathrm{d}^N l \frac{l_\mu l_\nu}{(l^2 + 2p \cdot l)^A} = \frac{\pi^{N/2}}{\Gamma(A) p^{2A-N}} \left[\Gamma(A - N/2) p_\mu p_\nu - \frac{1}{2} p^2 \delta_{\mu\nu} \Gamma(A - 1 - \frac{N}{2}) \right] \tag{3.38}$$

Here $\Gamma(x)$ is the Euler Gamma function. Using the identification $N = 2$, $A = 2$, $q = p/2$ we find that

$$\int \frac{\mathrm{d}^2 l}{2\pi} \frac{l_+(l_+ + q_+)}{l^2(l+q)^2} = -\frac{1}{4} \frac{q_+}{q_-} \tag{3.39}$$

Note that the term $\Gamma(A-1-N/2)$ in (3.38) will generate a term proportional to $\Gamma(0)$, which is undefined; in fact, $\Gamma(\epsilon)$ diverges logarithmically for $\epsilon \to 0$. Formally the loop diagram in figure 3.1 diverges, and the theory needs to be renormalized by absorbing the divergence in a field redefinition (see e.g. [6], chapter 7). For our purposes, we can neglect this term and only work with the finite part of the result.

Plugging this back into the conservation equation (3.32) we finally get the expectation of the trace of T_{ab} at one-loop order:

$$\langle T_{+-} \rangle = \frac{1}{4} \delta^{jk} R_{\mu jk\nu} \partial_a X_0^\mu \partial_b X_0^\nu \gamma^{ab} \tag{3.40}$$

This is the Weyl anomaly coefficient. We can contract the indices in the curvature tensor with the flat metric δ^{jk} to get the form

$$\langle T_{+-} \rangle = \frac{1}{4} R_{\mu\nu} \partial_a X_0^\mu \partial_b X_0^\nu \gamma^{ab} \tag{3.41}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor. We see that to first order, the Weyl anomaly coefficient depends on the curvature of spacetime. To recover Weyl invariance, at least at this order, we can impose the condition

$$R_{\mu\nu} = 0 \tag{3.42}$$

This is nothing but the Einstein field equations of General Relativity in vacuum. This is a remarkable result; we started with a general, Weyl invariant field theory in two dimensions with an arbitrary (albeit symmetric) target space metric $G_{\mu\nu}$. Quantizing the theory breaks Weyl invariance, and restoring the invariance imposes the requirement that $G_{\mu\nu}$ is in fact the metric tensor of a spacetime that obeys the vacuum field equations of General Relativity.

3.3 The Antisymmetric Tensor Action

In the previous section we only considered the Polyakov action S_P, with a symmetric, covariantly constant $G_{\mu\nu}$. We can ask whether there are any other terms that we can consider. The metric $G_{\mu\nu}$ in (2.1) is assumed to be a symmetric target space tensor, but there is no a priori reason to assume that the fields X cannot couple to an antisymmetric tensor as well. We can only add one term with an antisymmetric spacetime coupling that is both reparametrisation and Weyl invariant:

$$S_{AS} = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{3.43}$$

Here, $B_{\mu\nu} = -B_{\nu\mu}$ and ϵ^{ab} is the two-dimensional Levi Civita symbol. We can define a field strength H for the field $B_{\mu\nu}$ as

$$H_{\mu\nu\lambda} = \nabla_\mu B_{\nu\lambda} + \nabla_\nu B_{\lambda\mu} + \nabla_\lambda B_{\mu\nu} \tag{3.44}$$

If we now quantize the total action $S = S_P + S_{AS}$ along the lines of the previous section (see [1] for the lengthy, technical derivation), we find other contributions to the Weyl anomaly coefficients. Summarizing, to first order in perturbation theory (which includes first-order two-point correlators of the energy momentum tensor) there will be two independent Weyl anomaly coefficients, leading to two constraints on the target space tensors:

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = \frac{1}{4} \left[H_{\mu\lambda\sigma} H_\nu^{\lambda\sigma} - \frac{1}{6} G_{\mu\nu} H^2 \right] \tag{3.45}$$

$$\nabla^\lambda H_{\lambda\mu\nu} = 0 \tag{3.46}$$

Here, $R = G_{\mu\nu} R^{\mu\nu}$ is the Ricci scalar.

3.4 Consistency of the Weyl Anomaly Conditions

If we take a closer look at the anomaly conditions we notice, on the left hand side of the first equation (3.45), the spacetime Einstein tensor $R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R$. It is a symmetric tensor and it is covariantly conserved because of the Bianchi identity for the curvature tensor $R_{\mu\nu\rho\sigma}$. If the right hand side were to be conserved as well, this would be a consistent equation, and in fact would be the Einstein field equation in the presence of matter, with the right hand side representing the spacetime energy momentum tensor. That this is indeed the case can be checked immediately, and it is in fact the second condition (3.46) that insures that the spacetime stress energy tensor is conserved. The two equations (3.45) and (3.46) are therefore not purely independent but instead consistent with each other, and must hence be able to be derived as equations of motion from a certain action. The action, in D dimensions, whose variation reproduces the two Weyl anomaly conditions is

$$S_D = \int \mathrm{d}^D X \sqrt{G} \left\{ R - \frac{1}{12} H^2 \right\} \tag{3.47}$$

This is simpy the Einstein-Hilbert action along with a Maxwell type kinetic term for the antisymmetric tensor field. The conclusion we can draw then is that when we study the general non-linear sigma model in two dimensions, the requirement of maintaining Weyl invariance at the quantum level imposes on the fields and the coupling functions the structure of a D-dimensional gravitational spacetime whose dynamics are controlled by an extension of the Einstein-Hilbert action of general relativity. This remarkable conclusion has one main drawback, namely that it is only shown here to be true at one-loop order in perturbation theory. But one can, in principle, compute higher order contributions to the Weyl anomaly coefficients, and it is generally believed that to all orders the Weyl anomaly conditions can be derived from a master spacetime action. The exciting idea is that higher order contributions will add terms to the action which can be interpreted as string corrections to general relativity (at least in D dimensions).

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