Fibonacci anyons & Topological quantum Computers

By Christos Charalambous

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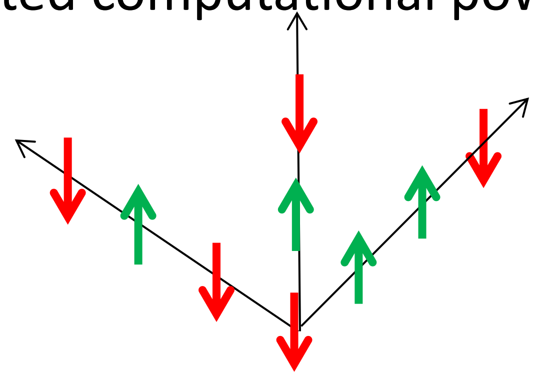
Quantum computation

- Classical computer: limited computational power



3d magnet

- Interference → Quantum physics could speed up processes

- Qubit: $|\psi> = \alpha|0> + \beta|1>$ where $a^2 + \beta^2 = 1, \quad a, \beta \in C$
  - Hilbert spaces: $\psi_1 \in H_1$, $\psi_2 \in H_2$

    → $\psi_{12} = \sum_{i,j=\{1,2\}} a_{ij} \psi_i \otimes \psi_j \in H_1 \otimes H_2$

    classical: m bits → 2^m states

    quantum: m qubits → 2^m **basis** states
  - Entanglement: speeding up classical algorithms

Quantum computation

- **Quantum Circuits:** unitary operators that act on a Hilbert space, generated by n qubits, whose states encode the information we want to process.

  Quantum Circuits are composed of elementary Quantum gates.

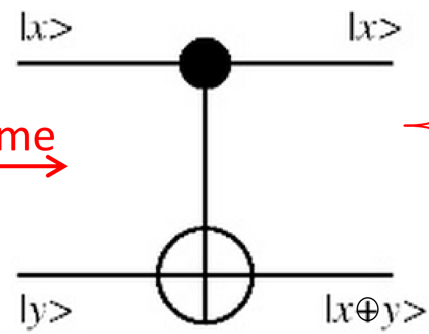
- **Universality**: existence of a universal set of quantum gates, the elements of which can perform any unitary evolution in SU(N) with arbitrary accuracy

  Need:

  1. Singe qubit rotation gates that can span SU(2): $|\psi\rangle \rightarrow U|\psi\rangle$
  2. A Two - qubit entangling gate: $U \in SU(2)$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

time →



x=0 → y unchanged

x=1 → $\begin{cases} y = 0 \rightarrow y = 1 \\ y = 1 \rightarrow y = 0 \end{cases}$

where ⊕ = addition mod 2

Examples of quantum algorithms: → Deutsch algorithm

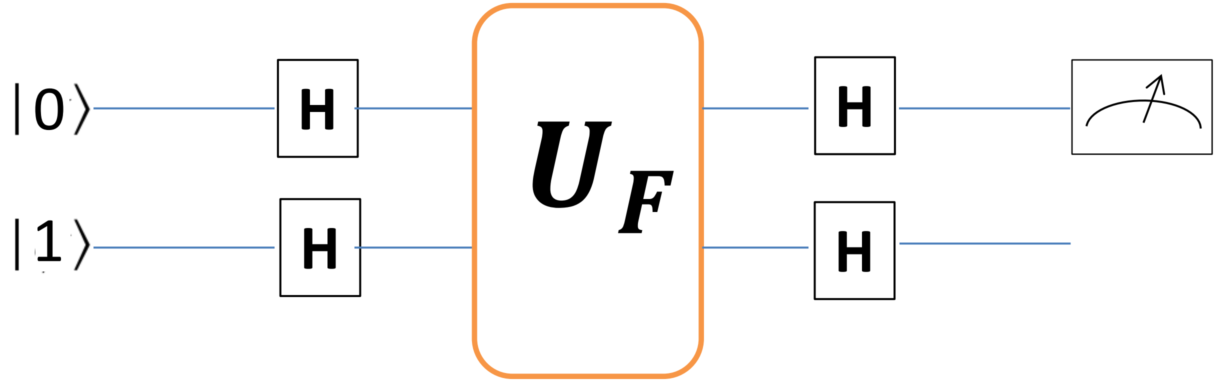
→ Shor's factoring algorithm

Quantum computation

- Deutsch Algorithm
  - Boolean Function F: Constant (f(0)=f(1)) or Balanced (f(0)≠f(1))?

Single qubit rotation gate called Hadamard :

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



  - Requires only 1 measurement to answer while classically it takes 2 evaluations of F

Decoherence

Very easy for errors to appear in the system due to interactions with the environment:

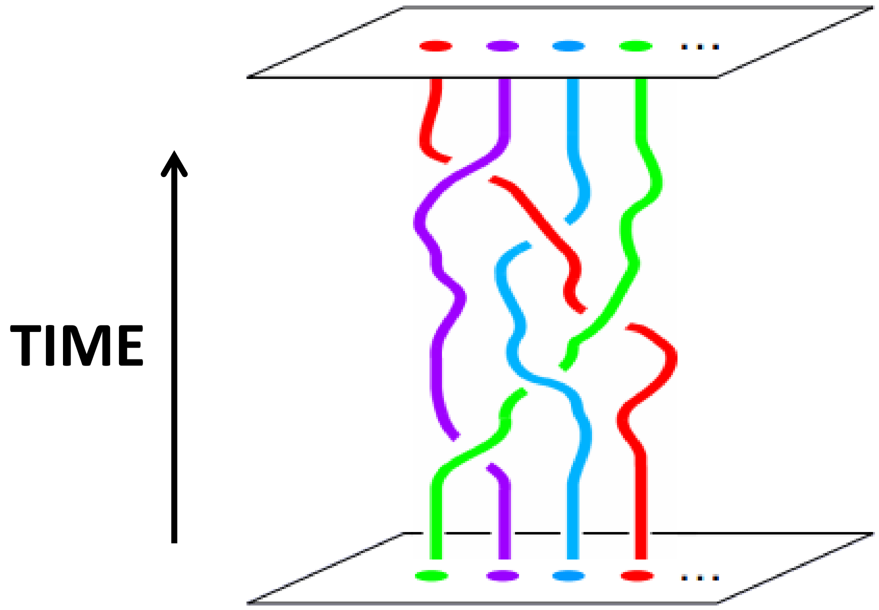
Examples: 1. Bit flips: $|0> \rightarrow |1>$, $|1> \rightarrow |0>$

2. Phase flips: $\frac{1}{\sqrt{2}}(|0>+|1>) \rightarrow \frac{1}{\sqrt{2}}(|0>-|1>)$

Quantum computation

Goal: encode information in an environment independent way

Idea: Topological properties are insensitive to local perturbations



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Abelian anyons

Exchanging:

$|\psi_A>|\psi_B> \;\rightarrow\; e^{i\theta}\,|\psi_B>|\psi_A>$

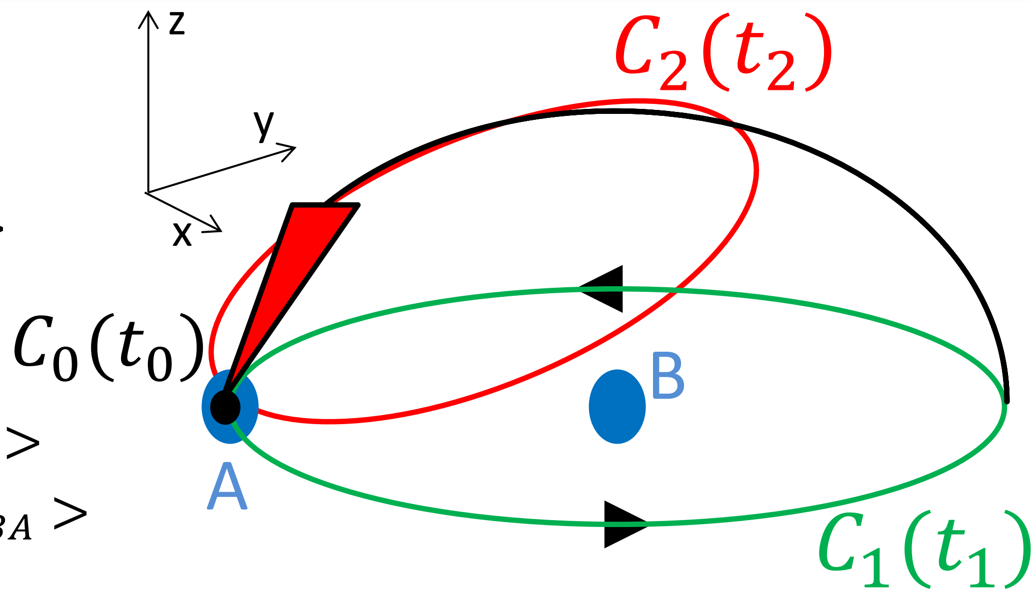
Winding:

$|\psi_A>|\psi_B> \;\rightarrow (e^{i\theta})^2\,|\psi_A>|\psi_B>$

In 3D: $(e^{i\theta})^2 = I$

→Boson: θ=2π+2πn $|\psi_{AB}> \rightarrow |\psi_{BA}>$

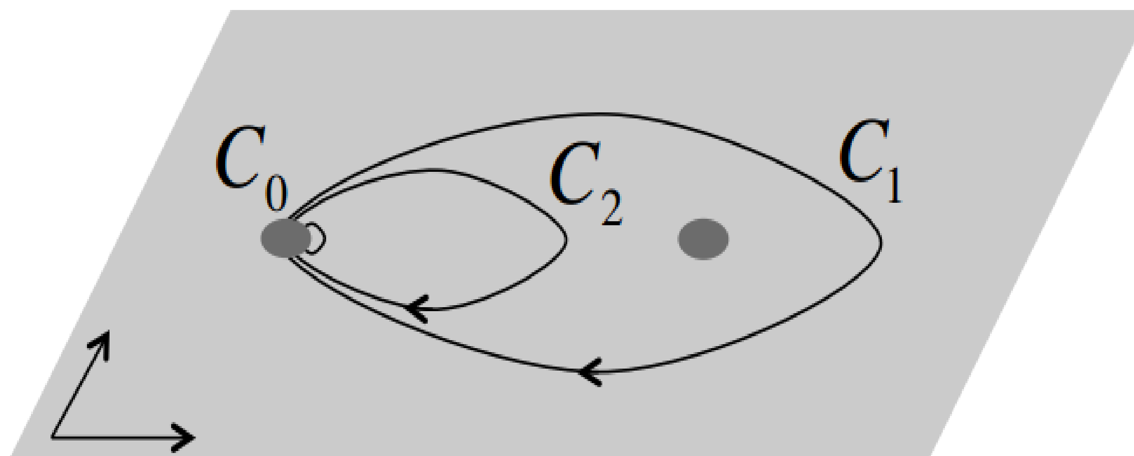
→Fermion: θ=π+2πn $|\psi_{AB}> \rightarrow -|\psi_{BA}>$



- Move to 2D:

Any θ → "Any"-ons

Exchanging=braiding



Non-abelian anyons

- Degenerate state space $\{\psi_i\}$, $i = 1, \ldots, d$ then:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix} \rightarrow \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \vdots \\ \psi'_d \end{pmatrix} = \begin{pmatrix} U_{11} & \cdots & U_{1d} \\ \vdots & \ddots & \vdots \\ U_{d1} & \cdots & U_{dd} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix}$$

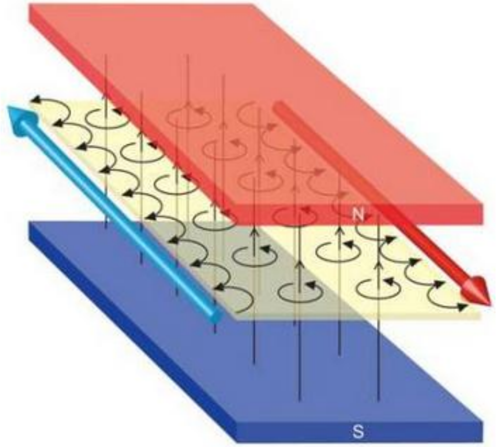
$\longrightarrow$ Candidate space to store and process quantum information

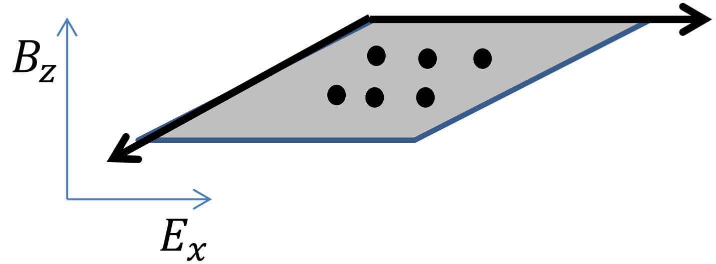
Physical realization of Non-abelian anyons

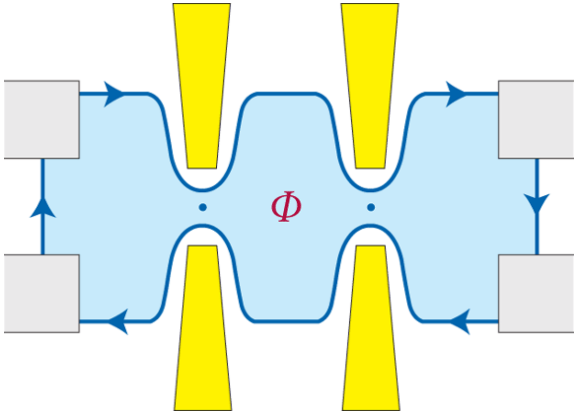
Physical realization of Non-abelian anyons:

1. Degenerate ground state
2. Finite energy gap ΔE for ground state
3. Adiabaticity
4. Anyons being far apart
5. All local operators have vanishing correlation functions apart from identity

Fractional Quantum Hall effect





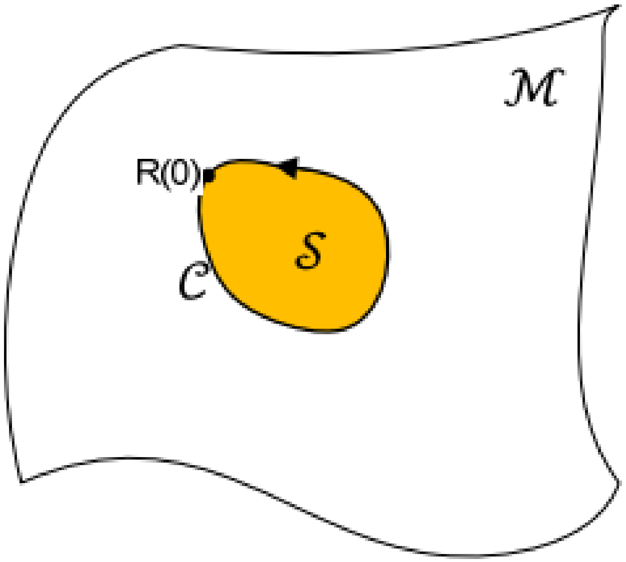


- Trapped e^- gas
- Conductance: $\sigma = \nu \frac{e^2}{h}$ where ν a fractional number
  → Fractional charge
- Abelian anyons for $\nu = \frac{1}{3}$

Laughlin states: Trial wavefunctions

Expect → fractional statistics = anyonic statistics

Geometric phase



- Time evolution of a state:

$$|\,\psi(\mathrm{T}) > = \exp\left(\mathrm{i}\gamma_n(C)\right) \exp\left\{\frac{-i2\pi}{h} \int_0^{\mathrm{T}} dt\, E_n\left(R(t)\right)\right\} |\,\psi(0) >$$

where $\gamma_n(C)$ is Berry's geometric phase:

$$\gamma_n(C) = i \oint_C \; < n, R(t) \,|\, \nabla_R n, R(t) > dR = \oint_C \; A_\mu dR^\mu = \oint_S \; \frac{1}{2} F_{\mu\nu} dR^\mu \wedge dR^\nu$$

Geometric phase

Vector potential gauge transformation:

$$A_\mu \rightarrow \mathrm{A}_\mu - \partial_\mu a_n$$

Vector field:

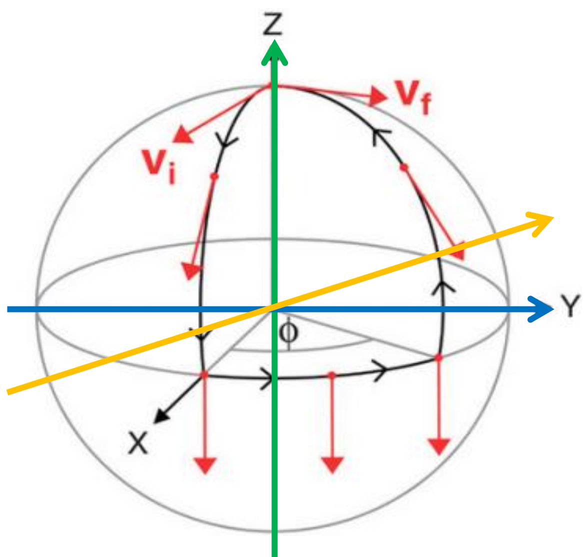
$$(F_{\mu\nu})^{if} \coloneqq (\partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu)^{if}$$

→ invariant under gauge transformations

If $(F_{\mu\nu})^{if} \neq 0$ i.e. not diffeomorphic invariant as well:

→ Case 1: Non-degenerate state space → Abelian geometric phase, U(1)

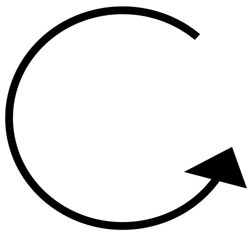
Example:

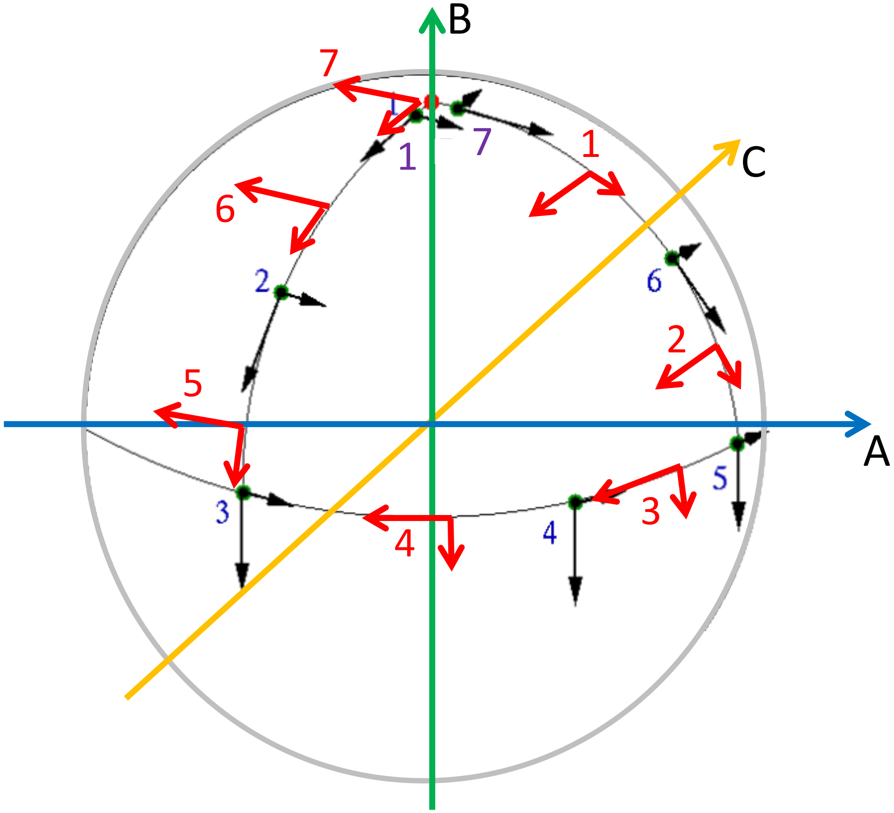


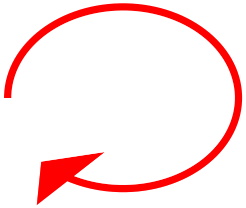
Geometric phase

→ Case 2: Degenerate state space ($(F_{\mu\nu})^{if}$ a matrix):

Example: $ABC \neq CBA$







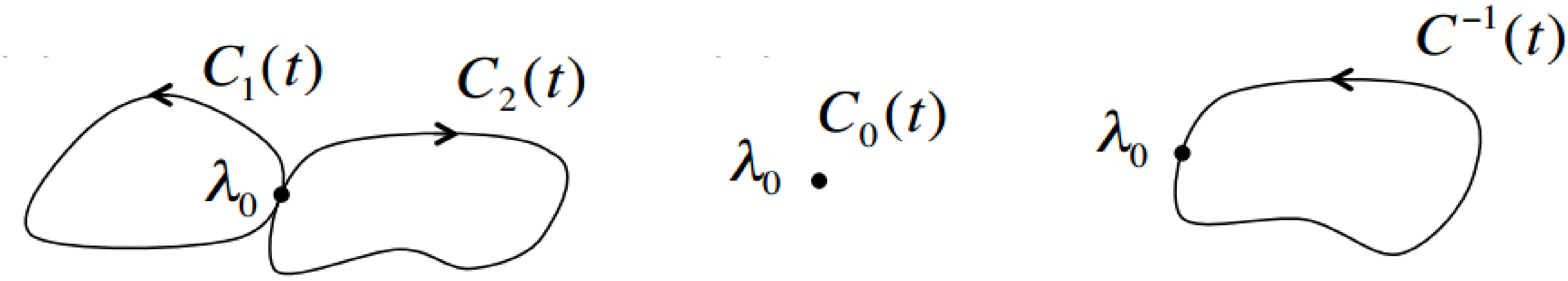
Transformations of the state space are elements of SU(2)

For N degenerate state space transformations are elements of SU(N)

Geometric phase

In general: $|\psi(C)> = \Gamma_A(C)\,|\psi(0)>$

$$\Gamma_A(C) = \exp\big(i\gamma_n(C)\big)$$



the following properties hold:

(A) $\Gamma_A(C_2 \cdot C_1) = \Gamma_A(C_2)\Gamma_A(C_1)$ — C_1, C_2 paths in parametric space

(B) $\Gamma_A(C_0) = I$ — C_0 point

(C) $\Gamma_A(C^{-1}) = \Gamma^{-1}{}_A(C)$ — C clockwise path, C^{-1} anti-clockwise path

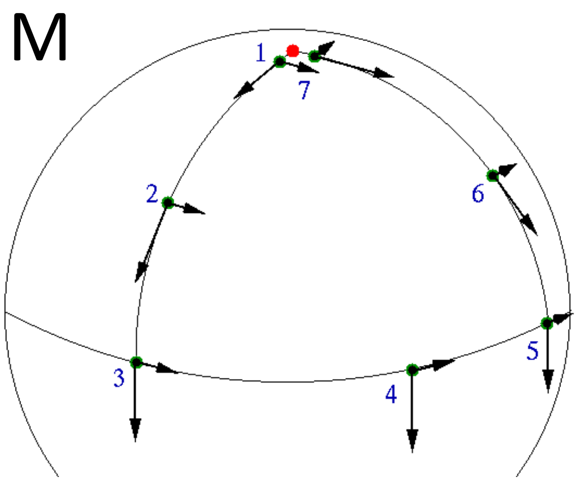
(D) $\Gamma_A(C \circ f) = \Gamma_A(C)$ — f is a function of time t

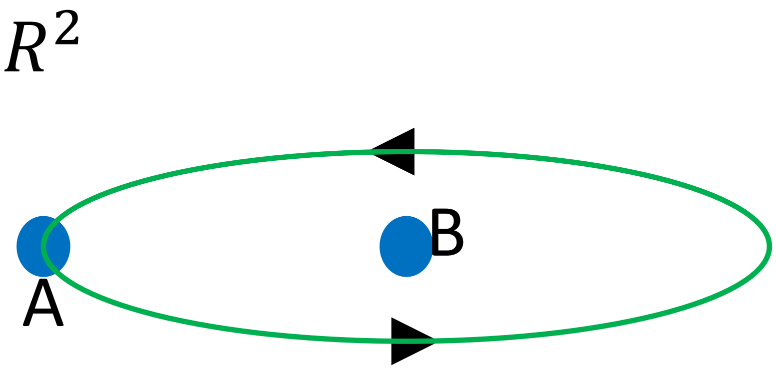
(A)+(B)+(C) → Forms a Group

Geometric phase

- Relate parametric space to anyons coordinates
- Assume vector field $F_{\mu\nu}$ is confined to anyons position

Hence:

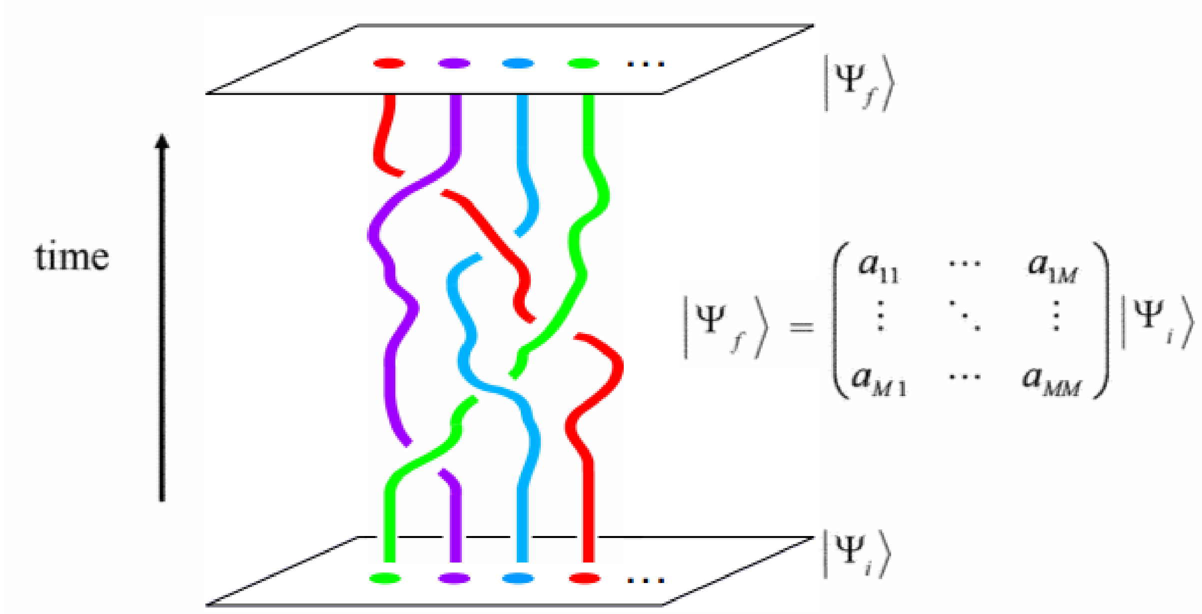




non-abelian geometric phases evolutions

evolutions in a system of non-abelian anyons

Topological Quantum Computer



- Quasiparticle worldlines forming braids carry out unitary transformations on a Hilbert space of n anyons.

- This Hilbert space is exponentially large and its states cannot be distinguished by local measurements

→ candidate model for fault-tolerant quantum computing

Topological Quantum Computer

- 1997 A.Kitaev: System of non-abelian anyons with suitable properties can efficiently simulate a quantum circuit

- 2000 Freedman, Kitaev and Wang: system of anyons can be simulated by a quantum circuit

→Equivalence of the two views of the system, i.e. between an anyonic computational model (e.g. a Topological quantum computer) and a quantum circuit

Is there an anyonic computational model that can simulate a quantum circuit that exhibits universality?

Contents

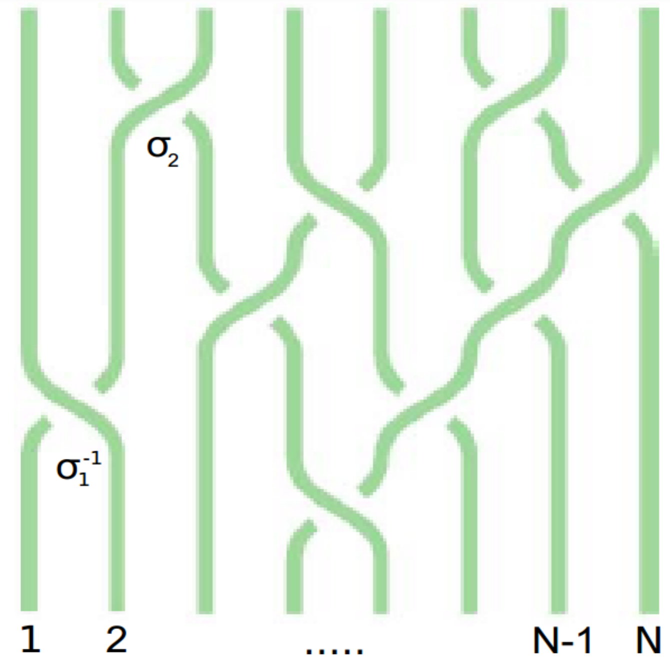
Braiding

$\sigma_i \coloneqq exchange\ of\ ith\ and\ (i+1)th\ anyon$

- World lines cannot cross

→ braids σ_i (particle histories) in distinct topological classes

- 1:1 correspondence of the topological classes with distinct elements of a Braid set



1. Any braid can be obtained by multiplying elementary braids
2. The inverse of any braid exists
3. Existence of Vacuum
4. Associativity for disjoint σ_i

→ **Braid group**

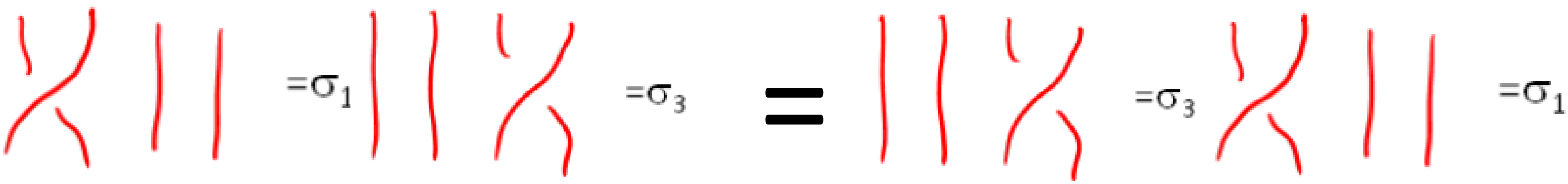
$\boldsymbol{\rho(\sigma_i)}$ **representations of the Braid group = the unitary transformations**

Braiding

Defining relations of Braid group for an anyonic model:

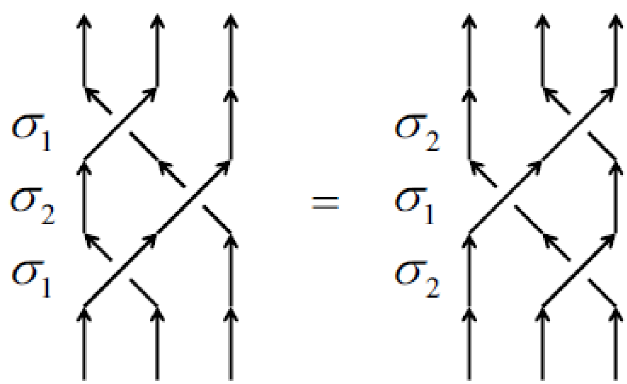
1. Exchanges of disjoint particles commute:

$$\sigma_j \sigma_k = \sigma_k \sigma_j \quad |\, j - k \,| \geq 2$$



2. Yang-Baxter relation:

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \quad j = 1,2, \dots, n-2$$



Fusion

- **Fusion**: The process of bringing two particles together

- A non abelian anyonic model is defined starting from the **superselection sector**:

Finite set of particles that are linked by the following fusion rules, and their charges are conserved under local operations

Fusion algebra: $a \times b = \sum_c N^c{}_{ab} c$

where $N^c{}_{ab}$ can be matrices

Abelian $N^c{}_{ab}$=1

Non-abelian $\sum_c N^c{}_{ab} > 1$

- Associativity: $\sum_e N^{\mathrm{e}}{}_{ab} N^c{}_{de} = \sum_e N^{\mathrm{e}}{}_{bd} N^c{}_{ea}$
- Commutativity: $N^c{}_{ab} = N^c{}_{ba}$

Recap: Fusion rules for minimal models

- Same fusion algebra:

$$\phi_i \times \phi_j = \sum_k {N^k}_{ij}\, \phi_k$$

- Commutativity also holds:

$${N^k}_{ij} = {N^k}_{ji}$$

- Associativity :

$$\sum_l {N^l}_{jk}\, {N^m}_{il} = \sum_l {N^l}_{ij}\, {N^m}_{lk}$$

→ Minimal models can be mapped to anyonic models

Fusion space

- Fusion spaces $V^c{}_{ab}$: are subspaces of the space of all possible fusion outcomes which are also **Hilbert spaces**:

$$V_a \otimes V_b = \oplus_c N^c{}_{ab} V^c{}_{ab}$$

The logical states $|\,0>$ & $|\,1>$ will be encoded in one of these fusion spaces

- Relation of dimensionality of fusion spaces and quantum dimension:

$$d_a d_b = \sum_c N^c{}_{ab} d_c$$

- Simplest non-abelian example: fusion rule for Spin-$\frac{1}{2}$ particles

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

(i.e. 2x2=1+3)

Fusion space basis (fusion trees)

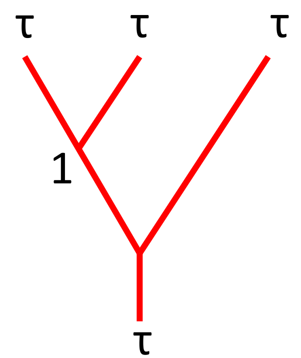
Fibonacci anyons fusion rule:

$$\tau \otimes \tau = 1 \oplus \tau$$

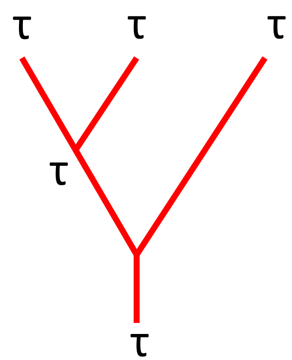
1 = vacuum

τ = non-abelian anyon

The Fusion (Anyonic) Hilbert space:



$$|\,(\tau\tau \rightarrow 1)\tau \rightarrow \tau >$$



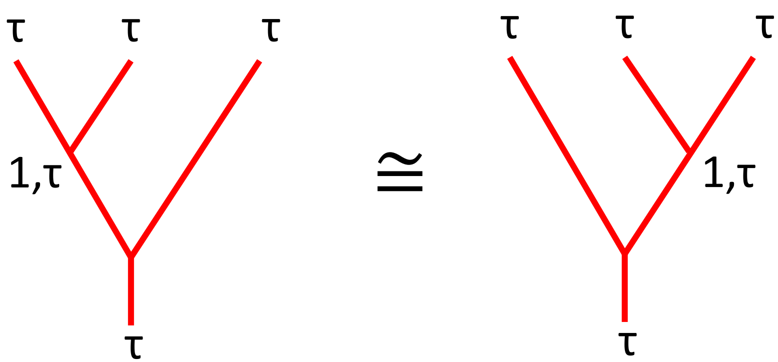
$$|\,(\tau\tau \rightarrow \tau)\tau \rightarrow \tau >$$

- Fusion trees are orthogonal basis elements of a Hilbert space
- If the initial and final states are fixed then the dimension of this space depends on the number of in-between outcomes. For the above example the dimension is 2.

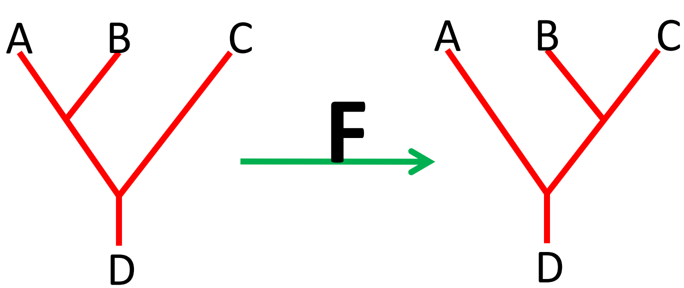
F & R matrices

The F-matrix

The order of the fusion should not be relevant (associativity):

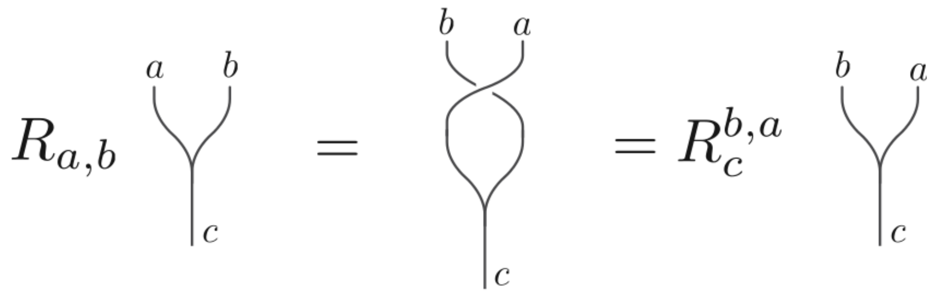


Therefore there exists a matrix F that transforms one basis to the other:

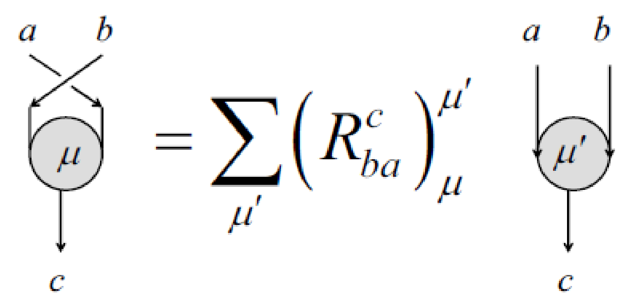


The R-matrix

Exchange of a & b before fusion= self-rotation of outcome c after fusion:



therefore just a phase factor is obtained. For many in between outcomes R = diagonal matrix:



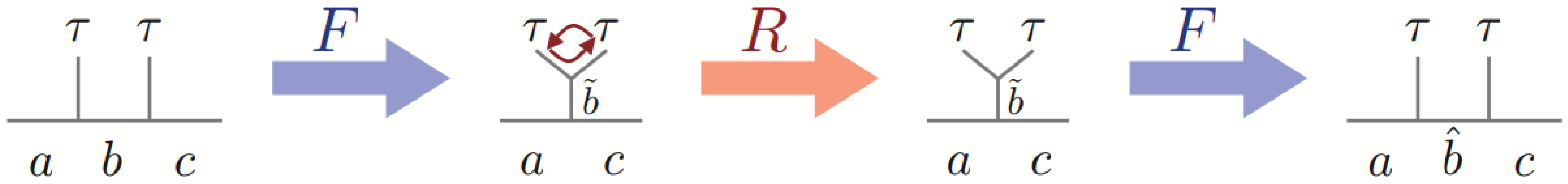
The unitary braiding matrix: B-matrix

- Consider superposition of multiple fusion outcomes, and consider the braiding of two particles that do no have a direct fusion outcome
  → exchanges result in a non-diagonal matrix R

- By applying F matrices on the R matrices we can change to a basis where the anyons do have a direct fusion outcome:

$$B = FRF^{-1} = \rho(\sigma_i)$$

Example:

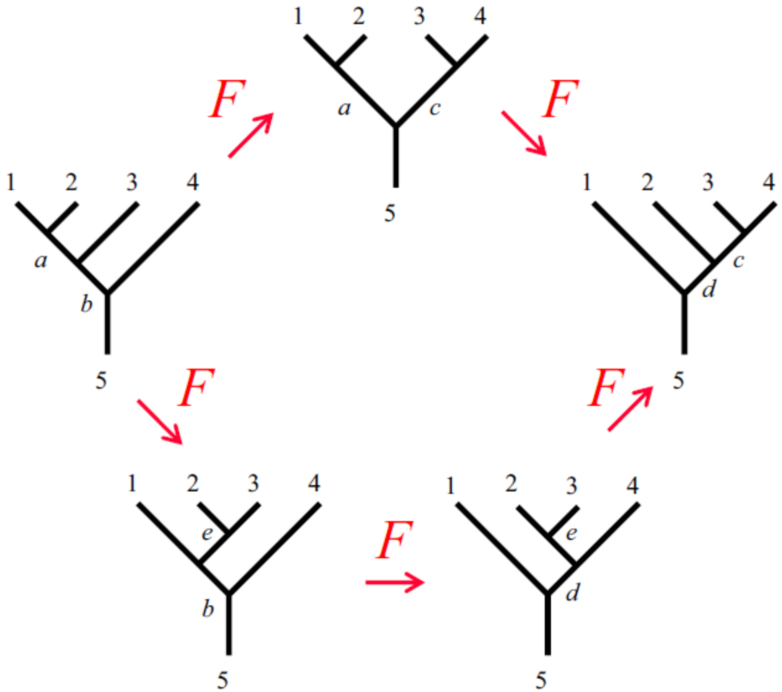


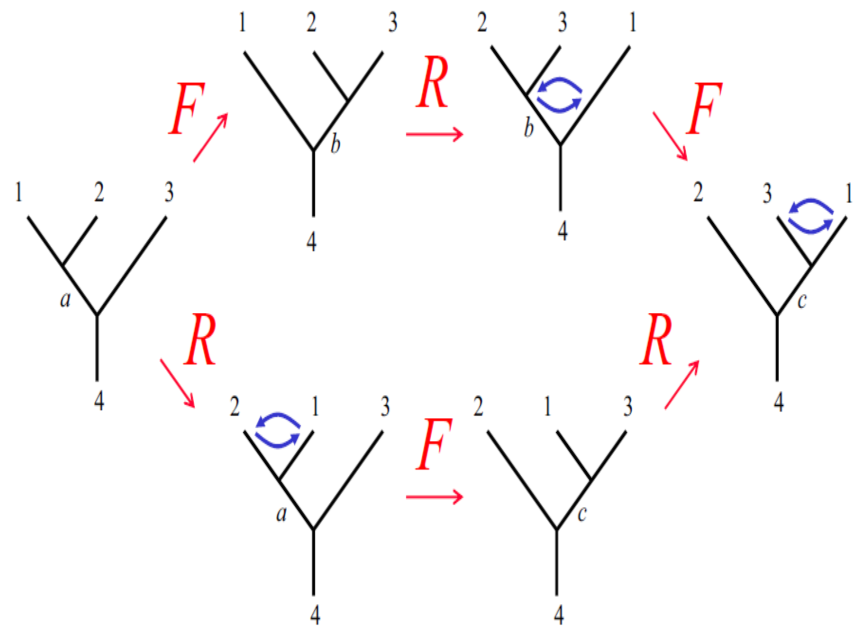
- F & R fully describe all the processes we can do in an anyonic model of computation

Compatibility equations: Pentagon & Hexagon equations

Fusion is associative → pentagon equation must hold:

Associativity of fusion + allow braiding → Hexagon equation





$$(F^{5}_{12c})^{d}_{a}(F^{5}_{a34})^{c}_{b} = \sum_{e}(F^{d}_{234})^{c}_{e}(F^{5}_{1e4})^{d}_{b}(F^{b}_{123})^{e}_{a}$$

$$\sum_{b}(F^{4}_{231})^{c}_{b}R^{4}_{1b}(F^{4}_{123})^{b}_{a} = R^{c}_{13}(F^{4}_{213})^{c}_{a}R^{a}_{12}$$

These two equations encode all the constraints we can impose on F & R

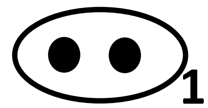
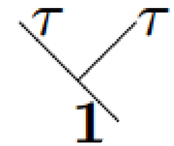
Contents

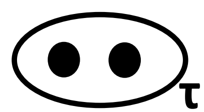
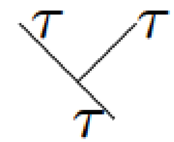
Fibonacci anyonic model

- Simple and rich structure

$$\tau \otimes \tau = 1 \oplus \tau$$

- Encoding of logical states (the naive way):

$|0\rangle = |(\bullet,\bullet)_1> =$  $=$ 

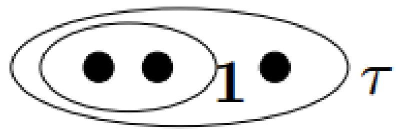
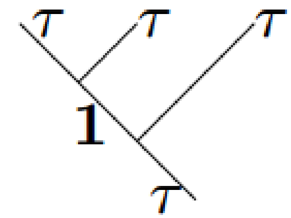
$|1\rangle = |(\bullet,\bullet)_\tau> =$  $=$ 

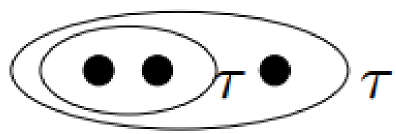
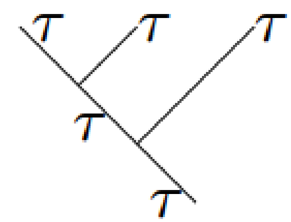
Fibonacci anyonic model

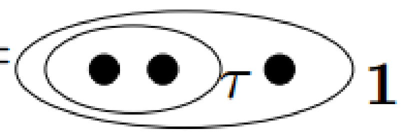
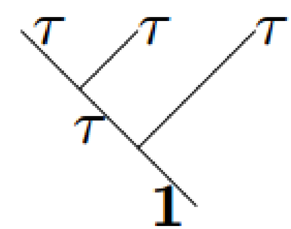
- Simple and rich structure

$$\tau \otimes \tau = 1 \oplus \tau$$

- Encoding of logical states:

$|0\rangle = |((\bullet, \bullet)_1, \bullet)_\tau\rangle =$  $=$ 

$|1\rangle = |((\bullet, \bullet)_\tau, \bullet)_\tau\rangle =$  $=$ 

$|N\rangle = |((\bullet, \bullet)_\tau, \bullet)_1\rangle =$  $=$ 

The last state is not a problem as we will see right now:

Fibonacci anyonic model

From fusion rules and pentagon equation for Fibonacci model:

$$F^{\tau\tau\tau}{}_{1} = F^{1\tau\tau}{}_{\tau} = F^{\tau 1\tau}{}_{\tau} = F^{\tau\tau 1}{}_{\tau} = 1$$

$$F^{\tau\tau\tau}{}_{\tau} = \begin{pmatrix} \frac{1}{\varphi} & \frac{1}{\sqrt{\varphi}} \\ \frac{1}{\sqrt{\varphi}} & -\frac{1}{\varphi} \end{pmatrix}$$

where ϕ is the golden ratio $\varphi = \frac{(1+\sqrt{5})}{2}$

From hexagon equation and Yang-baxter relation:

$$R^{\tau 1}{}_{\tau} = R^{1\tau}{}_{\tau} = 1$$

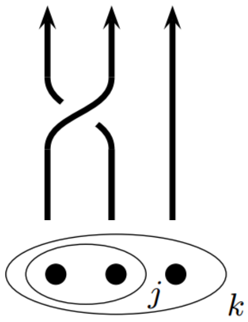
$$R^{\tau\tau} = \begin{pmatrix} e^{i4\pi/5} & 0 \\ 0 & -e^{i2\pi/5} \end{pmatrix}$$

Fibonacci anyonic model

Braiding matrices obtained from F and R:

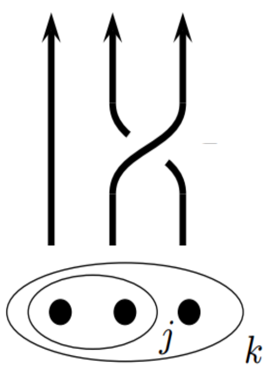
$$\begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \rightarrow \underbrace{\left(\begin{array}{cc|c} e^{-4\pi i/5} & 0 & 0 \\ 0 & -e^{-2\pi i/5} & 0 \\ \hline 0 & 0 & -e^{-2\pi i/5} \end{array}\right)}_{\rho(\sigma_1)} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix}$$

$\sigma_1 =$



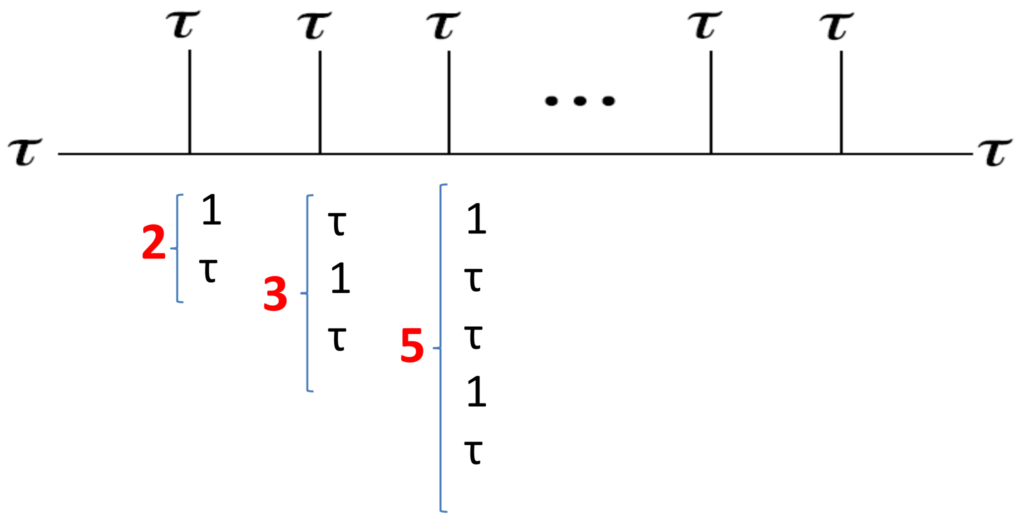
$$\begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \rightarrow \underbrace{\left(\begin{array}{cc|c} -e^{-\pi i/5}/\phi & -ie^{-i\pi/10}/\sqrt{\phi} & 0 \\ -ie^{-i\pi/10}/\sqrt{\phi} & -1/\phi & 0 \\ \hline 0 & 0 & -e^{-2\pi i/5} \end{array}\right)}_{\rho(\sigma_2)} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix}$$

$\sigma_2 =$

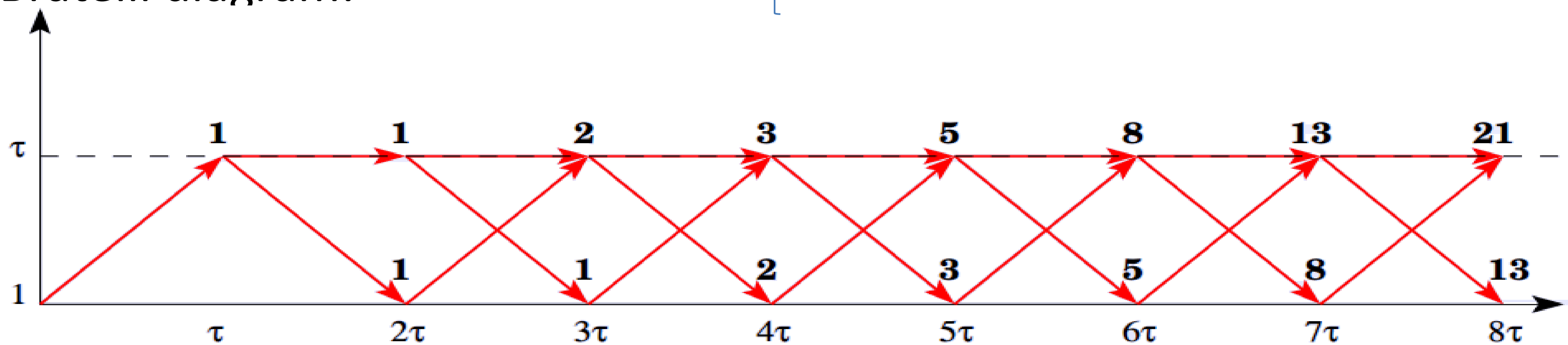


Fibonacci anyonic model

Dimension of Hilbert space for Fibonacci model with the constraint that no two consecutive 1's can appear



Bratelli diagram:



→ Dimension of the space is $\propto \Phi^n$

Fibonacci anyonic model

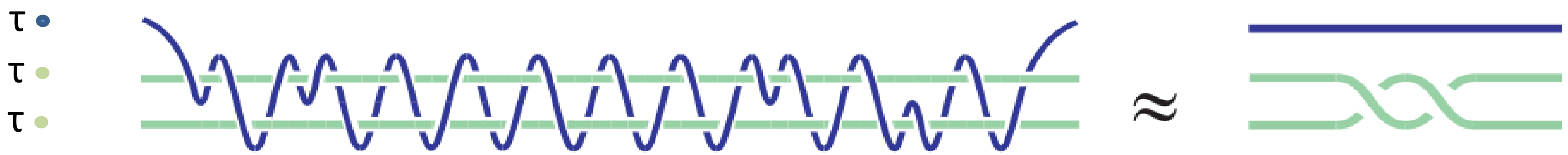
Universality

- Solovay and Kitaev (version of brute force search algorithm):

Combine short braids → can obtain a long braid that with **arbitrary accuracy ε** will simulate a desired **single qubit unitary operation**

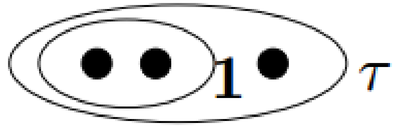
- Bonesteel, Hormozi: **2-qubit entangling gate CNOT**

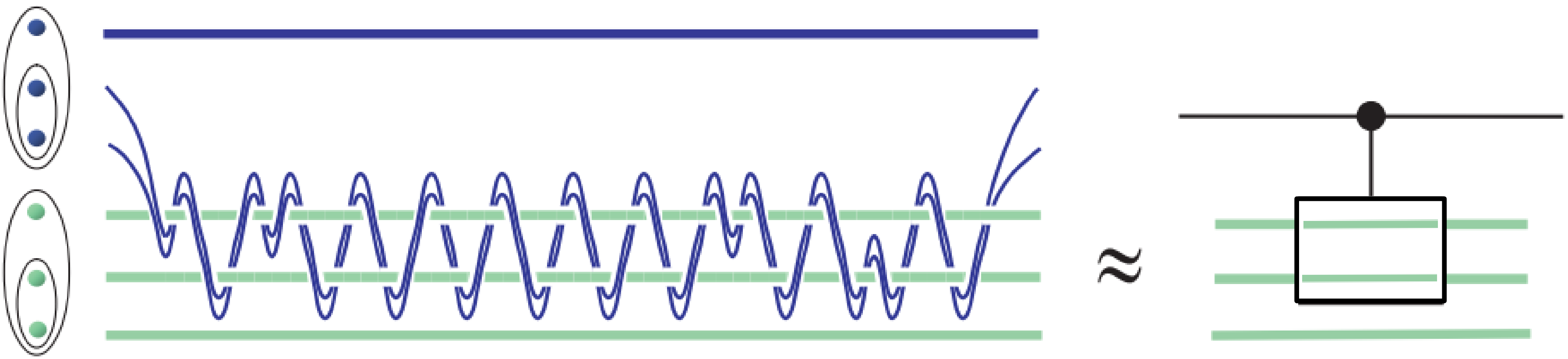
1st observation:

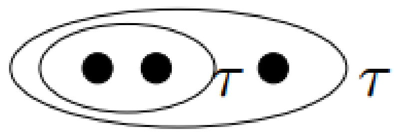


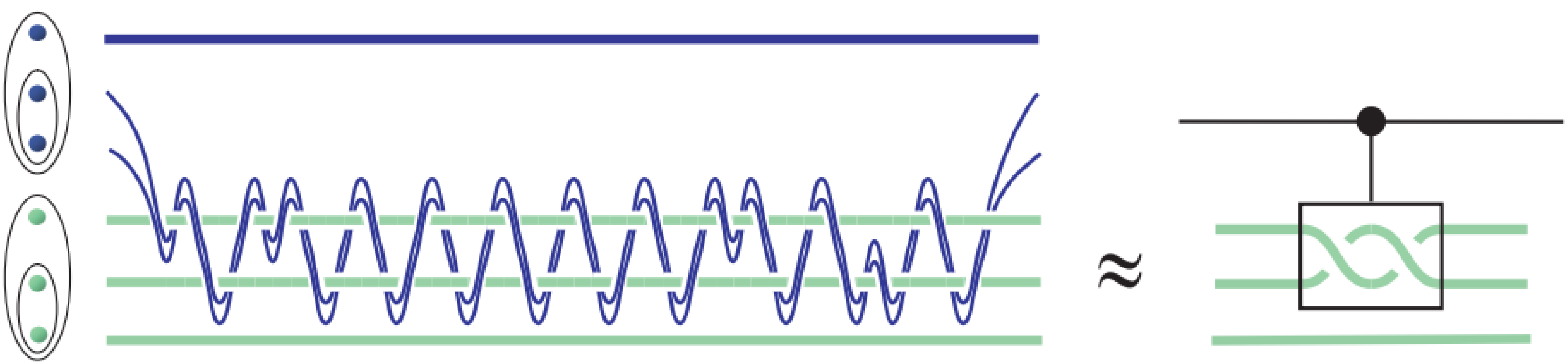
Fibonacci anyonic model

2nd observation:

Case 1: upper qubit is $|0\rangle =$ 



Case 2: upper qubit is $|1\rangle =$ 



Fibonacci anyonic model

→ $\rho(\sigma_1)$ & $\rho(\sigma_2)$ acting on the logical states can perform any unitary evolution in SU(N)

→ **universal computations**

Conclusion:

Fibonacci anyonic model:

- Can achieve universal computing
- well-controlled accuracy
- Requires 4n physical anyons for encoding n logical qubits (i.e. polynomial scaling)

Fibonacci anyons and $SU(2)_3$

- spin-1 particles: $1 \otimes 1 = 0 \oplus 1 \oplus 2$
- Similarity to: $\tau \otimes \tau = 0 \oplus \tau$ if spin-2 is cut off

- $SU(2)_k$: " quantized" version of SU(2) obtained by truncating the possible values of the angular momentum to

$$j = 0, \frac{1}{2}, 1, \frac{3}{2} \dots, \frac{k}{2}$$

e.g. $SU(2)_3 = \{0, \frac{1}{2}, 1, \frac{3}{2}\}$

- Consider only 0 & 1 particles of $SU(2)_3$

→subgroup ("even" part) of $SU(2)_3 \cong$ superselection sector of Fibonacci model

- A model described by such symmetry is the $\boldsymbol{SU(2)_3}$ **WZW model** coupled to a U(1) gauge field

Non-abelian Chern-Simons Theory

$$S_{CS}(A) = \frac{k}{4\pi}\int_{\mathrm{M}} d^3x\varepsilon^{\mu\nu\rho}\, tr\left(A_\mu \partial_\nu A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right) = \frac{1}{4\pi}\int_{\mathrm{M}} d^3x L_{CS}(A)$$

where k: coupling constant , $M = \Sigma \times \mathrm{R}$ (2+1D) , A is a gauge field

- No metric → invariant under diffeomorphisms
- Non Abelian Gauge transformations:

$$A'_\mu = gA_\mu g^{-1} - ig\partial_\mu g^{-1} \quad \text{where} \quad g: M \to G$$

$$\rightarrow L_{CS}(A') =$$

$$L_{CS}(A) - k\varepsilon^{\mu\nu\rho}\partial_\mu tr\left((\partial_\nu g)g^{-1}A_\rho\right) - \frac{k}{3}\varepsilon^{\mu\nu\rho} tr\left(g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\right)$$

For suitable boundary conditions the 1st extra term vanishes in the action.

Non-abelian Chern-Simons Theory

In the case of simple compact groups e.g. G=SU(2):

$$\omega(g) = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho} tr\left(g^{-1}(\partial_\mu g)g^{-1}(\partial_\nu g)g^{-1}(\partial_\rho g)\right)$$

where **ω(g)=winding number** and we realize that is proportional to the 2nd extra term in the Lagrangian. Hence the action becomes:

$$\rightarrow S_{CS}(A') = S_{CS}(A) + 2\pi\kappa\omega(g)$$

- If ω(g)=0 (small gauge transformations, low energies)

  → action is invariant

- If ω(g)≠0 (large gauge transformations, high energies)

  → failure of gauge invariance

Case 1: k integer → OK

Case 2: k not an integer → **gapped pure gauge degrees of freedom for the high energy theory**

Non-abelian Chern-Simons Theory

$$H = \frac{k}{4\pi} tr(A_2 \partial_0 A_1 - A_1 \partial_0 A_2) - L = 0$$

Easily seen if we choose gauge $A_0 = 0$ where the momenta canonically conjugated to: $A_1 : -\frac{k}{4\pi} A_2$, $A_2 : \frac{k}{4\pi} A_1$

Introduce spatial boundaries $M = \partial\Sigma \times \mathrm{R}$

- Locally (bulk part): Gauge invariance
- BUT globally: topological obstruction in making the gauge field zero everywhere if Σ topologically non-trivial

→ **Chern-Simons gauge invariant up to a surface term**

→ **physical topological degrees of freedom**

→ **Chern-Simons: theory of ground state of a 2D topologically ordered system in Σ**

Non-abelian Chern-Simons Theory

Conclusion

- For low energies:

→ Chern Simons theory describes non abelian anyons.

- For high energies:

→ Difficult to disentangle physical topological degrees of freedom from unphysical local gauge degrees of freedom

→ hence have to consider Chern Simons as low energy effective field theory

What is the theory that describes the excitations of these quasiparticles?

Recap of WZW models

- WZW models describe Symmetry Protected topological Phases (SPT) in 2D at an open boundary with symmetry SU(2).
- Showed global and local SU(2) invariance of J_{+} (the WZW charge carrying covariant current density) coupled with an external field action
- Showed that integrating action with an external field leads to an effective field theory:

WZW action coupled with external field at low energies

$\cong$ Chern Simon action

For WZW models:

→ k= number of anyon species in the theory → integer

→ gapless WZW gauge degrees of freedom = CS pure gauge degrees of freedom

→Solved problem of Chern-Simons in higher energies

Summary

1. Defined Quantum Computer and identified problem of decoherence

2. Identified topological properties as a remedy for the problem

3. Identified anyons as systems that exhibit such topological properties and hence under specific conditions can accomodate a Topological quantum computer

4. Examined Fibonacci anyons as candidate particles for performing Universal Topological quantum computing

5. Showed that such particles can be theoretically described in the context of a CFT model