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Proseminar in CFT and String Theory

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D-branes

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Abstract

This work pretends to give a basic view of the main aspects of bosonic string theory, T-duality and D-branes. We take as starting point the Nambu-Goto action and from there proceed to find an adequate parameterization so that we can solve the equation of motion of both open and closed strings. Next, quantization is discussed in some detail for both open and closed strings. T-duality for closed strings is sketched and D-branes are introduced as a preamble to be able to understand T-duality in the presence of open strings. Finally some particular D-branes configurations with background electromagnetic fields are studied and related to their dual equivalent and the Born-Infeld lagrangian is briefly discussed.

Emphasis is made on how string theory is able to reproduce and describe concepts that appear in quantum field theory such as Yang-Mills theories. The approach to solve the equations of motion and to quantize the theory is made in the light-cone gauge and using light-cone coordinates.

The major omissions are superstrings and the covariant approach in terms of the Polyakov action.

Contents

1 The Nambu-Goto action and its symmetries

1.1 Nambu-Goto action

The departure point for bosonic strings is the Nambu-Goto action. It is a direct generalization of the action of a relativistic point particle since it is basically the volume of the world-sheet traced by the string, the only sensible geometrical quantity.

The two parameters that characterize the non-relativistic string are the tension and the mass density. In the case of the relativistic string there is only one characteristic parameter, which we will take to be the tension T_0. The canonical speed in the theory of relativity is c, the speed of light, it is therefore logical to propose the following action functional based only upon Lorentz symmetry and dimensional analysis considerations:

$$S = \frac{-T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \tag{1}$$

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \qquad\qquad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma} \tag{2}$$

1.2 Four-momentum conservation

The Nambu-Goto action only depends on derivatives of the string coordinates therefore it is invariant under variations $\delta X^\mu(\tau, \sigma) = \epsilon^\mu$, i.e. under a constant translation in spacetime. In this circumstances Noether theorem guarantees the existence of a conserved current. We know that translational invariance is related to four-momentum conservation so we expect the conserved charge to be the four-momentum.

The conserved current is:

$$j_\mu^\alpha = \frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha X^\mu\right)} \equiv \mathcal{P}_\mu^\alpha \tag{3}$$

The components of the conserved current are just the conjugate momenta, the conservation law and the conserved charge are:

$$\partial_\alpha \mathcal{P}_\mu^\alpha = \frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0 \tag{4}$$

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}_\mu^\tau(\tau, \sigma) d\sigma \tag{5}$$

The integral is to be performed with τ held constant. The conserved currents live on the world-sheet and so we should be able to obtain the charges using more general curves, in fact we have:

$$p_\mu = \int_\gamma \left(\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau\right) \tag{6}$$

for any curve γ with endpoints on the boundary of the world-sheet.

1.3 Lorentz symmetry

The Nambu-Goto action is also invariant under Lorentz transformations: $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$. Lorentz invariance is related to angular momentum conservation. Applying Noether theorem we obtain:

$$\epsilon^{\mu\nu} j^\alpha_{\mu\nu} = \left(-\frac{1}{2}\epsilon^{\mu\nu}\right)\left(X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu\right) \tag{7}$$

The conserved current is:

$$\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu \tag{8}$$

The conservation law and the conserved charges are:

$$\partial_\alpha \mathcal{M}^\alpha_{\mu\nu} = \frac{\partial \mathcal{M}^\tau_{\mu\nu}}{\partial\tau} + \frac{\partial \mathcal{M}^\sigma_{\mu\nu}}{\partial\sigma} = 0 \tag{9}$$

$$M_{\mu\nu} = \int_\gamma \left(\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau\right) \tag{10}$$

1.4 The slope parameter α'

The string tension T_0 is the only dimensionful parameter in the string action. However it is more usual to work with the alternative parameter α'. The parameter α' has an interesting physical interpretation, used since the early days of string theory when it was tought as a theory of hadrons. Consider a rigidly rotating open string, then α' is the proportionality factor that relates the angular momentum J of the string measured in terms of $\hbar$, to the square of its energy E.

$$\frac{J}{\hbar} = \alpha' E^2 \tag{11}$$

Assume the string is rotating in the (x, y) plane, then $J = |M_{12}|$. We will just quote the result:

$$J = \left| \int_0^{\sigma_1} \left(X_1 \mathcal{P}^\tau_2 - X_2 \mathcal{P}^\tau_1\right) \right| = \frac{\sigma_1^2 T_0}{2\pi c} \tag{12}$$

Since $\sigma_1 = \frac{E}{T_0}$ we obtain:

$$\alpha' = \frac{1}{2\pi T_0 \hbar c} \tag{13}$$

2 Equations of motion, boundary conditions, and D-branes

The dynamics of the string is obtained as usual by varying the action:

$$\delta S = 0 \tag{14}$$

$$\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial}{\partial \tau} \left(\delta X^\mu \mathcal{P}^\tau_\mu \right) + \frac{\partial}{\partial \sigma} \left(\delta X^\mu \mathcal{P}^\sigma_\mu \right) - \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} \right) \right] = 0 \tag{15}$$

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_\tau X^\mu \right)} = -\frac{T_0}{c} \frac{\left(\dot{X} \cdot X' \right) X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X' \right)^2 - \left(\dot{X} \right)^2 (X')^2}} \tag{16}$$

$$\mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_\sigma X^\mu \right)} = -\frac{T_0}{c} \frac{\left(\dot{X} \cdot X' \right) \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{\left(\dot{X} \cdot X' \right)^2 - \left(\dot{X} \right)^2 (X')^2}} \tag{17}$$

The first term can be ignored if we restrict ourselves to variations for which $\delta X^\mu(\tau_f, \sigma) = \delta X^\mu(\tau_i, \sigma) = 0$. The second term has to do with the string endpoints and is in fact a collection of $2D = 2(d+1)$ terms, we need appropiate boundary conditions to make all these terms vanish. The last term must vanish for all δX^μ so it must be:

$$\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0 \tag{18}$$

This is the equation of motion of the relativistic string, note that it coincides with the four-momentum conservation law derived in section 1.2. The equation of motion is really complicated, to be able to solve it we will have to cast it in a simpler form using reparameterization invariance, before turning our attention to that point lets say a few words about the possible boundary conditions.

Let σ_* be an endpoint then there are two natural options to make any of the single terms vanish:

$$\textit{Dirichlet} \quad \textit{b.c.} \qquad \delta X^\mu(\tau, \sigma_*) = 0 \Longrightarrow \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \mu \neq 0 \tag{19}$$

$$\textit{Free endpoint b.c.} \qquad \mathcal{P}^{\sigma}_{\mu}(\tau,\sigma_*) = 0 \tag{20}$$

The second condition is called a free endpoint condition because it does not impose any constraint on the variation $\delta X^{\mu}(\tau,\sigma_*)$, the endpoint is free to do whatever is needed to get the variation of the action to vanish. Time must flow as τ flows so for $\mu = 0$ the only possibility is to impose free endpoint condition $\mathcal{P}^{\sigma}_{0}(\tau,\sigma_1) = \mathcal{P}^{\sigma}_{0}(\tau,0) = 0$. Eventually we will understand the free endpoint b.c as a Neumann b.c.

These boundary conditions can be imposed in many possible ways. For each spatial direction, and at each endpoint we can choose either a Dirichlet or a free endpoint b.c. Since closed strings have no endpoints, they do not require boundary conditions.

When we study the dynamics of a non-relativistic string we impose Dirichlet or Neumann b.c. depending on whether the endpoints are attached or free to move. The objects on which open string endpoints must lie are characterized by the number of spatial dimensions that they have. They are called D-branes,the letter D stands for Dirichlet. A Dp-brane is an object with p spatial dimensions. Since the string endpoints mut lie on the Dp-brane, a set of Dirichlet boundary conditions is specified. D-branes are not necessarily hyperplanes nor are they necessarily of infinite extent, even though we will only consider this simple case. When the open string endpoints satisfy free boundary conditions along all spatial direction, we still have a D-brane, but this time it is a space-filling D-brane.

At this stage we can point out an "inconsistency" of the theory. Previously we derived the conservation of four-momemtum p^{μ} however:

$$\frac{dp^{\mu}}{d\tau} = \int_0^{\sigma_1} \frac{\partial \mathcal{P}^{\tau}_{\mu}}{\partial \tau} d\sigma = -\int_0^{\sigma_1} \frac{\partial \mathcal{P}^{\sigma}_{\mu}}{\partial \sigma} d\sigma = -\mathcal{P}^{\sigma}_{\mu} \,|_0^{\sigma_1} \tag{21}$$

The right hand side vanishes in the case of closed strings since the points $0, \sigma_1$ are to be identified. It also vanishes for open strings with free endpoints boundary conditions but is not necessarily vanishing for open strings with Dirichlet boundary conditions so we may wonder if this boundary conditions are appropiate at all. The way out of this difficulty is to recognise that the D-branes where the open string endpoints are attached are dynamical objects in their own right, in fact one can write down an action functional for D-branes and check that they can carry energy, momentum and they can even be charged. In this work we treat D-branes passively, we never give them dynamical character by writing down the action, however is important to keep in mind these considerations to avoid conceptual difficulties in the development. In this particular situation the problem disappears when we acknowledge that the D-brane being a dynamical object can carry the momentum that flows off the string endpoint and so as a whole momentum is conserved.

3 The static gauge

3.1 Static gauge

The equation of motion of the string is very complicated, to be able to solve it we will need to exploit reparameterization invariance and simplify the equation. In this section we introduce a partial parameterization that will be useful to gain some insight.

We fix the lines of constant τ by relating τ to the time coordinate $X^0 = ct$ in some chosen Lorentz frame. Consider the hyperplane $t = t_0$ in the target space. This plane will intersect the world-sheet along a curve-the string at time t_0 according to observers in our chosen Lorentz frame. We declare this curve to be the curve $\tau = t_0$. Extend this definition for all times.

We do not try at the moment to make any smart choice of σ. For an open string we just require one edge of the world sheet to be the curve $\sigma = 0$ and the other edge to be the curve $\sigma = \sigma_1$ i.e. $\sigma \in [0, \sigma_1]$

In this gauge:

$$\frac{\partial X^\mu}{\partial \sigma} = \left(0, \frac{\partial \vec{X}}{\partial \sigma}\right) \qquad \frac{\partial X^\mu}{\partial \tau} = \left(c, \frac{\partial \vec{X}}{\partial t}\right) \tag{22}$$

3.2 String action in terms of transverse velocity

The first thing we will do is to rewrite the Nambu-Goto action in a way where the similarities with the relativistic point particle is more manifest. We could define the string velocity as $\frac{\partial \vec{X}}{\partial t}$, however this quantity is gauge dependent, different parameterizations of σ would yield different concepts of velocity, this suggest that longitudinal motion on the string is not physically meaningful.

There exists a velocity that is gauge independent, the transverse velocity:

$$\vec{v}_\perp \equiv \frac{\partial \vec{X}}{\partial t} - \left(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s}\right) \frac{\partial \vec{X}}{\partial s} \tag{23}$$

where s is a parameter that measures length along the string, i.e. $ds = |d\vec{X}| = |\frac{\partial \vec{X}}{\partial \sigma}||d\sigma|$ therefore $\frac{\partial \vec{X}}{\partial s}$ is a unit vector tangent to the string.

It is not hard to show that the argument of the square root in the Nambu-Goto action can be written as:

$$\sqrt{\left(\dot{X}\cdot X'\right)^2-\left(\dot{X}\right)^2(X')^2}=c\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}} \tag{24}$$

So:

$$S=-T_0\int dt\int_0^{\sigma_1}d\sigma\left(\frac{ds}{d\sigma}\right)\sqrt{1-\frac{v_\perp^2}{c^2}} \tag{25}$$

The Lagrangian is then $L=-T_0\int ds\sqrt{1-\frac{v_\perp^2}{c^2}}$, since $T_0 ds$ can be identified with the rest energy is clear what this formula is telling us: to build the Lagrangian of a relativistic string just add up all the Lagrangians of the infinitesimal relativistic segments but be aware that in each infinitesimal segment only transverse motion is relevant. Now the analogy with the point particle action is clear, also there is no reason to doubt about the minus sign that appears in the Nambu-Goto action which could at first look somehow suspicious.

3.3 Open string endpoints

Now we use the static gauge to obtain some information about how the endpoints move. We will show that:

- The endpoints move transversely to the string.

In the static gauge we have:

$$\begin{aligned}\mathcal{P}^{\sigma\mu}&=\frac{-T_0}{c}\frac{\left(\frac{\partial\vec{X}}{\partial\sigma}\cdot\frac{\partial\vec{X}}{\partial t}\right)\dot{X}^\mu-\left(-c^2+\left(\frac{\partial\vec{X}}{\partial t}\right)^2\right)X'^\mu}{c\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}}}=\\&=\frac{-T_0}{c^2}\frac{\left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)\dot{X}^\mu+\left(c^2-\left(\frac{\partial\vec{X}}{\partial t}\right)^2\right)\frac{\partial X^\mu}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\end{aligned} \tag{26}$$

In particular we have that $\mathcal{P}^{\sigma 0}=\frac{-T_0}{c}\frac{\left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)}{\sqrt{1-\frac{v_\perp^2}{c^2}}}$. As we mentioned before the time coordinate must satisfy free endpoint boundary condition so $\mathcal{P}^{\sigma 0}(\tau,\sigma_*)=0$ and therefore we conclude $\left(\frac{\partial\vec{X}}{\partial s}\cdot\frac{\partial\vec{X}}{\partial t}\right)=0$ at the endpoints.

- The endpoints move with the speed of light.

In view of our previous result we can write:

$$\mathcal{P}^{\sigma\mu}(\tau, \sigma_*) = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial X^\mu}{\partial s} = 0; \quad \mu = 1, 2, ..., d \tag{27}$$

or

$$\vec{\mathcal{P}}^{\sigma}(\tau, \sigma_*) = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial \vec{X}}{\partial s} = 0 \Longrightarrow v = c \tag{28}$$

4 Light cone gauge

4.1 Gauge fixing

In this section we present a class of gauges to parameterize the world-sheet, then the equation of motion is finally solved.

Consider the equation $n_\mu x^\mu(\tau, \sigma) = \lambda\tau$, where λ is some constant with units of velocity. This is the equation of a hyperplane in the target space, if two points x_1, x_2 satisfy the equation then $n_\mu(x_1^\mu - x_2^\mu) = 0$ i.e. the vector that joins the two point is orthogonal to the vector n^μ. The points X^μ that satisfy $n_\mu X^\mu = \lambda\tau$ are points that lie both on the world-sheet and on the hyperplane. All these points must be assigned the same value τ.

We need strings to be spacelike objects, perhaps null in some limit, but certainly never timelike. A timelike n^μ is enough to guarantee this, but it is too restrictive, we will allow both timelike and null. None of this gauges is Lorentz covariant.

In section 1 we showed that due to translational invariance there is a conserved charge p^μ, we use this Lorentz vector to rewrite our gauge condition as follows:

$$n \cdot X(\tau, \sigma) = \tilde{\lambda}(n \cdot p)\tau \tag{29}$$

Since $n \cdot p$ is a constant the net effect is that we have traded the constant λ for another constant $\tilde{\lambda}$. When open strings are attached to D-branes not all components of the string momentum are conserved. Since we want our analysis to hold even in this case, we will assume that the vector n^μ is chosen in such a way that $n \cdot p$ is conserved. This condition is weaker than the condition of momentum conservation. We will assume that $n \cdot \mathcal{P}^\sigma = 0$ at the open string endpoints since this condition naturally guarantees the conservation of $n \cdot p$.

By involving n^μ on both sides of the equation, the length of n^μ has been made irrelevant. Only the direction of n^μ matters. This new constant has dimensions of $\tilde{\lambda} \sim \frac{c}{T_0} = 2\pi\alpha'\hbar c^2$.

From now on we work in natural units and write the action in terms of α' rather than T_0.

The Nambu-Goto action in this units is

$$S = \frac{-1}{2\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2} \tag{30}$$

We choose $\tilde{\lambda} = 2\alpha'$ for open strings.

So far for the τ parameterization. The σ parameterization is fixed by requiring the constancy of $n_\mu \mathcal{P}^{\tau\mu}$ over the strings. Additionally, we require a parameterization range $\sigma \in [0, \pi]$ for open strings or $\sigma \in [0, 2\pi]$ for the case of closed strings. We can in fact do so since in the static gauge:

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2} \frac{ds}{d\sigma} \frac{\dot{X}^\mu - \left(\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t}\right) \frac{\partial X^\mu}{\partial s}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \tag{31}$$

Since all the σ dependence comes from the fraction $\frac{ds}{d\sigma}$ if we are given a parameterization $\tilde{\sigma}$ in which $n \cdot \mathcal{P}^\tau(\tau, \tilde{\sigma})$ depends on $\tilde{\sigma}$, we can choose a σ parameter so that $n \cdot \mathcal{P}^\tau(\tau, \sigma)$ does not depend on σ. A simple rescaling now will yield the appropiate range for σ. In this final parameterization we have:

$$n \cdot \mathcal{P}^\tau(\tau, \sigma) = a(\tau) \tag{32}$$

Where $a(\tau)$ is some function of τ. In fact $a(\tau)$ is already fixed by the conditions we have imposed:

$$\int_0^\pi d\sigma\, n \cdot \mathcal{P}^\tau(\tau, \sigma) = n \cdot p = \pi a(\tau) \Longrightarrow a(\tau) = \frac{n \cdot p}{\pi} \tag{33}$$

Dotting the equation of motion with n^μ we obtain $\frac{\partial}{\partial\sigma}\left(n \cdot \mathcal{P}^\sigma\right) = 0$. This means that $n \cdot \mathcal{P}^\sigma$ is constant along the string. We required that $n \cdot \mathcal{P}^\sigma = 0$ at the open string endpoints and so $n \cdot \mathcal{P}^\sigma = 0$ all along the string.

For closed strings we want a range $\sigma \in [0, 2\pi]$ so in this case $n \cdot \mathcal{P}^\tau = \frac{n \cdot p}{2\pi}$, due to this change is convenient to choose $\tilde{\lambda} = \alpha'$ for closed strings so the gauge fixing is $n \cdot X = \alpha'(n \cdot p)\tau$.

In resume:

$$n \cdot X(\tau, \sigma) = \beta\alpha'(n \cdot p)\tau \tag{34}$$

$$n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau \tag{35}$$

$$n \cdot \mathcal{P}^\sigma = 0 \tag{36}$$

Where it is understood that $\beta = 2$ for open strings and $\beta = 1$ for closed strings. Note that we did not prove (36) for closed strings. For open strings we know that $n \cdot \mathcal{P}^\sigma = 0$ at the endpoints however in the case of closed strings there are no special points were $n \cdot \mathcal{P}^\sigma = 0$ is known to vanish. In addition there is not a natural way to select the point $\sigma = 0$ at each value of τ. It turns out that the two problems can be solved at once and there is enough freedom so that (36) can be consistenly imposed.

4.2 Constraints due to the gauge fixing

There are some constraints in $\dot{X}$ and X' implied by our choice of parameterization:

$$0 = n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'} \frac{\left(\dot{X} \cdot X'\right) \partial_\tau \left(n \cdot X\right) - \left(\dot{X}\right)^2 \partial_\sigma \left(n \cdot X\right)}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}} \tag{37}$$

In view of (34), (37) implies $\dot{X} \cdot X' = 0$, this result together with (34) and (35) gives $n \cdot p = \frac{1}{\beta\alpha'} \frac{X'^2\left(n \cdot \dot{X}\right)}{\sqrt{-\dot{X}^2 X'^2}} \Longrightarrow \dot{X}^2 + X'^2 = 0$.

These two constraints can be conveniently resumed as:

$$\left(\dot{X} \pm X'\right)^2 = 0 \tag{38}$$

Taking into account the constraints equation the conjugate momenta have simple expressions:

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu \tag{39}$$

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu'} \tag{40}$$

In view of (40) we can say that free endpoint boundary conditions are just Neumann boundary conditions, as we commented in section 2.

4.3 Wave equation and solution

Applying the constraints the equation of motion simplifes to a simple wave equation:

$$\ddot{X}^\mu - X''^\mu = 0 \tag{41}$$

The most general solution to the wave equation is:

$$X^\mu(\tau, \sigma) = \frac{1}{2} \left(f^\mu(\tau + \sigma) + g^\mu(\tau - \sigma)\right) \tag{42}$$

Lets say that we want to study the case of an open string in a space filling D-brane, in this case all the coordinates satisfy free endpoint boundary conditions:

$$\mathcal{P}^{\sigma\mu}(\tau,\sigma=0)=\frac{\partial X^\mu}{\partial\sigma}(\tau,\sigma=0)=\frac{1}{2}\left(f'^{\mu}(\tau)-g'^{\mu}(\tau)\right)=0 \tag{43}$$

So:

$$X^\mu(\tau,\sigma)=\frac{1}{2}\left(f^\mu(\tau+\sigma)+f^\mu(\tau-\sigma)\right) \tag{44}$$

The boundary condition at the other endpoint yields:

$$\mathcal{P}^{\sigma\mu}(\tau,\sigma=\pi)=\frac{\partial X^\mu}{\partial\sigma}(\tau,\sigma=\pi)=\frac{1}{2}\left(f'^{\mu}(\tau+\pi)-f'^{\mu}(\tau-\pi)\right)=0 \tag{45}$$

So f'^{μ} is 2π periodic and therefore it admits a Fourier expansion:

$$f'^{\mu}(u)=f_1^\mu+\sum_{n=1}^{\infty}\left(a_n^\mu\cos(nu)+b_n^\mu\sin(nu)\right) \tag{46}$$

The solution to the wave equation is then:

$$X^\mu(\tau,\sigma)=f_0^\mu+f_1^\mu\tau+\sum_{n=1}^{\infty}\left(A_n^\mu\cos(n\tau)+B_n^\mu\sin(n\tau)\right)\cos(n\sigma) \tag{47}$$

The constant f_1^μ is related to momentum since $p^\mu=\int_0^\pi\mathcal{P}^{\tau\mu}d\sigma=\frac{1}{2\pi\alpha'}f_1^\mu$. We can make the Fourier expansion in terms of exponentials rather than sine and cosine:

$$X^\mu(\tau,\sigma)=x_0^\mu+2\alpha' p^\mu\tau-i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(a_n^{\mu *}e^{in\tau}-a_n^\mu e^{-in\tau}\right)\frac{\cos(n\sigma)}{\sqrt{n}} \tag{48}$$

We introduce some new notation that will prove convenient:

$$\alpha_0^\mu\equiv\sqrt{2\alpha'}p^\mu \qquad \alpha_n^\mu\equiv a_n^\mu\sqrt{n} \qquad \alpha_{-n}^\mu\equiv a_n^{\mu *}\sqrt{n} \tag{49}$$

We can write the solution to the wave equation in this new notation:

$$X^\mu(\tau,\sigma)=x_0^\mu+\sqrt{2\alpha'}\alpha_0^\mu\tau+i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma) \tag{50}$$

We have found solutions to the wave equation that satisfy the appropiate boundary conditions, but we must also make sure that the constraints $\left(\dot{X}\pm X'\right)^2=0$ are satisfied. If we specify arbitrarily all the constants α_n^μ, the constraints will not be satisfied. In the next section we use the light-cone gauge to find solutions that also satisfy the constraints.

4.4 Light-cone solution of equations of motion

We select the light-cone gauge by chosing $n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0\right)$ then:

$$X^+ \equiv n \cdot X = \beta\alpha' p^+ \tau \tag{51}$$

$$p^+ \equiv n \cdot p = \frac{2\pi}{\beta}\mathcal{P}^{\tau +} \tag{52}$$

The rest of coordinates $X^2, ..., X^d$ are called transverse coordinates and are generally denoted by X^I. The constraint equations in this gauge read:

$$-2\left(\dot{X}^+ \pm X'^+\right)\left(\dot{X}^- \pm X'^-\right) + \left(\dot{X}^I \pm X'^I\right)^2 = 0 \tag{53}$$

$$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'}\frac{1}{2p^+}\left(\dot{X}^I \pm X'^I\right)^2 \tag{54}$$

Once the transverse coordinates are specified, X^- is known up to a constant of integration (a zero mode). X^+ has a simple form due to our choice of gauge so the dynamics of the string is almost entirely contained in the transverse coordinates, this makes sense since in section 3.2 we realized that longitudinal motion was irrelevant and only transverse velocity naturally appears in the action. The full evolution of the string is therefore determined by the following set of objects:

$$X^I(\tau, \sigma), \quad p^+, \quad x_0^- \tag{55}$$

The mode expansion of a transverse coordinate is just as (50):

$$X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I \tau + i\sqrt{2\alpha'}\sum_{n \neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau}\cos(n\sigma) \tag{56}$$

By our choice of gauge:

$$X^+(\tau, \sigma) = 2\alpha' p^+ \tau = \sqrt{2\alpha'}\alpha_0^+ \tau \tag{57}$$

The zero mode and the oscillations of the X^+ coordinate have been set to zero by the gauge fixing:

$$x_0^+ = 0 \qquad \alpha_n^+ = \alpha_{-n}^+ = 0 \qquad n = 1, 2, ..., \infty \tag{58}$$

The X^- coordinate being a linear combination of X^0 and X^1 has a similar wave expansion:

$$X^-(\tau, \sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^- \tau + i\sqrt{2\alpha'}\sum_{n \neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos(n\sigma) \tag{59}$$

We find by direct calculation:

$$\dot{X}^- \pm X^{-'} = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^- e^{-in(\tau\pm\sigma)} \tag{60}$$

$$\dot{X}^I \pm X^{I'} = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-in(\tau\pm\sigma)} \tag{61}$$

Replacing (60) and (61) in (54) we find:

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+} \sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I \tag{62}$$

So a full solution is specified by the values of:

$$p^+, \qquad x_0^-, \qquad x_0^I, \qquad \alpha_n^I \tag{63}$$

The quadratic combination of oscillators on the right-hand side of (62) is a very important quantity so it has been given its own name. It is the transverse Virasoro mode $L_n^\perp$:

$$L_n^\perp \equiv \frac{1}{2} \sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I \tag{64}$$

For $n = 0$ and recalling that $\alpha_0^\mu \equiv \sqrt{2\alpha'}p^\mu$ we find:

$$\sqrt{2\alpha'}\alpha_0^- = 2\alpha' p^- = \frac{1}{p^+} L_0^\perp \Longrightarrow 2p^+ p^- = \frac{1}{\alpha'} L_0^\perp \tag{65}$$

The mass of the string is given by:

$$M^2 = -p^2 = 2p^+p^- - p^I p^I = \frac{1}{\alpha'} \sum_{n=1}^{\infty} n a_n^{I*} a_n^I \tag{66}$$

This classical result will not survive quantization. First, M^2 will become quantized, and string states will not exhibit a continuous spectrum of masses. Furthermore quantum mechanics will shift the formula by a constant.

5 Relativistic quantum open strings

5.1 Light-cone commutators

To quantize the classical theory we must impose appropiate equal-time commutation relations among the coordinates and their conjugate momenta. The Heisenberg operators are:

$$X^I(\tau,\sigma), \quad x_0^-(\tau), \quad \mathcal{P}^{\tau I}(\tau,\sigma), \quad p^+(\tau) \tag{67}$$

The only non-trivial commutation relations are:

$$\left[X^I(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')\right]=i\eta^{IJ}\delta(\sigma-\sigma') \qquad \left[x_0^-(\tau),p^+(\tau)\right]=-i \tag{68}$$

5.2 Commutation relations for oscillators

It is convenient to recast the commutation relations in terms of oscillators, to acccomplish this it will be useful to have at hand the following result:

$$\begin{aligned}&\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\pm X'^J\right)(\tau,\sigma')\right]=\\&=\pm\left[\dot{X}^I(\tau,\sigma),X'^J(\tau,\sigma')\right]\pm\left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]=\\&=\pm4\pi\alpha' i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma')\end{aligned} \tag{69}$$

Also we can find:

$$\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\mp X'^J\right)(\tau,\sigma')\right]=0 \tag{70}$$

Lets define:

$$A^I(\tau,\sigma)\equiv\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I e^{-in(\tau+\sigma)}=\begin{cases}\left(\dot{X}^I+X'^I\right)(\tau,\sigma) & \sigma\in[0,\pi]\\ \left(\dot{X}^I-X'^I\right)(\tau,-\sigma) & \sigma\in[-\pi,0]\end{cases} \tag{71}$$

Using (69),(70) and (71) we find:

$$\left[A^I(\tau,\sigma),A^J(\tau,\sigma')\right]=4\pi\alpha' i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma'), \qquad \sigma,\sigma'\in[-\pi,\pi] \tag{72}$$

So:

$$\sum_{m',n'\in\mathbb{Z}}e^{-im'(\tau+\sigma)}e^{-in'(\tau+\sigma')}\left[\alpha_{m'}^I,\alpha_{n'}^J\right]=2\pi i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{73}$$

The trick to extract only one term out of the sum is to perform in both sides the integration $\frac{1}{2\pi}\int_0^{2\pi}d\sigma e^{im\sigma}\frac{1}{2\pi}\int_0^{2\pi}d\sigma' e^{in\sigma'}$:

$$\left[\alpha_m^I,\alpha_n^J\right]=m\eta^{IJ}\delta_{m+n,0} \tag{74}$$

In terms of a_m^I and $a_n^{J\dagger}$ the commutation relations are:

$$\left[a_m^I,a_n^{J\dagger}\right]=\frac{m}{\sqrt{mn}}\delta_{m,n}\eta^{IJ}=\delta_{m,n}\eta^{IJ} \tag{75}$$

These are the typical communtation relations of a harmonic oscillator, we recognise:

$$\begin{aligned} &\alpha_n^I \qquad \textit{are annihilation operators} \\ &\alpha_{-n}^I \qquad \textit{are creation operators } (n \geq 1) \end{aligned} \tag{76}$$

To complete the list of non-commuting objects we need to find the commutator of x_0^I and α_n^J. Integrate expresion (68) in $\int_0^\pi d\sigma$ to get:

$$\left[x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau, \dot{X}^J(\tau,\sigma')\right] = 2\alpha' i\eta^{IJ} = \sqrt{2\alpha'} i\eta^{ij} \tag{77}$$

$$\sum_{n'\in\mathbb{Z}} \left[x_0^I, \alpha_{n'}^J\right] cos(n'\sigma')e^{-in'\tau} = \sqrt{2\alpha'} i\eta^{IJ} \tag{78}$$

Integrate this relation over $\frac{1}{\pi}\int_0^\pi d\sigma cos(n\sigma)$ with $n \geq 1$ to obtain:

$$\left[x_0^I, \alpha_n^J e^{-in\tau} + \alpha_{-n}^J e^{in\tau}\right] = 0 \tag{79}$$

This is only possible if both commutators vanish separately thus we find $\left[x_0^I, \alpha_n^J\right] = 0$ for $n \neq 0$.

If $n = 0$ then we obtain $\left[x_0^I, \alpha_0^J\right] = \sqrt{2\alpha'} i\eta^{IJ}$. So as expected $\left[x_0^I, p^J\right] = i\eta^{IJ}$.

5.3 Transverse Virasoro mode

We obtained the mode expansion of the string coordinates X^μ and constructed the mass operator before quantizing the theory. Upon quantization some results become ambiguous due to ordering issues, so now that we have a quatum theory we should solve these ambiguities.

Lets look at the transverse Virasoro modes defined in section 4.4: $L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p^I}\alpha_p^I$, back there the oscillators were just numbers but in the quantum theory they are non-commuting operators so is not clear which order is the appropiate one. In fact the only ambiguous operator is L_0.

$$L_0^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}^I\alpha_p^I = \frac{1}{2}\alpha_0^I\alpha_0^I + \frac{1}{2}\sum_{p=1}^{\infty} \alpha_{-p}^I\alpha_p^I + \frac{1}{2}\sum_{p=1}^{\infty} \alpha_p^I\alpha_{-p}^I \tag{80}$$

There is no doubt about how to order the first term on the right hand side. The second term is normal-ordered (creation operators appear to the left of annihilation operators), it is useful to have all our operators normal-ordered so following this criteria we reorder the third term in normal order:

$\frac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I = \frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty}p$ and so:

$$L_0^{\perp} = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \frac{1}{2}(D-2)\sum_{p=1}^{\infty}p \tag{81}$$

This expression is problematic because the last term is divergent but in fact is not as crazy as it seems, introducing a regulator and understanding $\sum_{p=1}^{\infty}p$ as the analytic continuation of the Riemman zeta function we have the result $\sum_{p=1}^{\infty}p = -\frac{1}{12}$. As we will show later this reasoning provides the right answer.

Lets define once and for all $L_0^{\perp}$ as:

$$L_0^{\perp} \equiv \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I = \alpha' p^I p^I + \sum_{p=1}^{\infty}p a_p^{I\dagger}a_p^I \tag{82}$$

And lets just call the reordering constant a. As it turns out the consistency of the quantum theory will fix the value of a.

The mass operator is now modified as well:

$$M^2 = \frac{1}{\alpha'}\left(a + \sum_{n=1}^{\infty}n a_n^{I\dagger}a_n^I\right) \tag{83}$$

5.4 Lorentz generators

The conserved charges due to Lorentz invariance were previously computed:

$$\begin{aligned}
M^{\mu\nu} &= \int_0^{\pi}\mathcal{M}^{\tau\mu\nu}(\tau,\sigma)d\sigma = \int_0^{\pi}\left(X^{\mu}\mathcal{P}^{\tau\nu} - X^{\nu}\mathcal{P}^{\tau\mu}\right)d\sigma = \\
&= \frac{1}{2\pi\alpha'}\int_0^{\pi}\left(X^{\mu}\dot{X}^{\nu} - X^{\nu}\dot{X}\right) \Longrightarrow \\
M^{\mu\nu} &= x_0^{\mu}p^{\nu} - x_0^{\nu}p^{\mu} - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^{\mu}\alpha_n^{\nu} - \alpha_{-n}^{\nu}\alpha_n^{\mu}\right)
\end{aligned} \tag{84}$$

The third term of the Lorentz generators is quadratic in the oscillators. In the quantum theory these oscillators are to be understood as operators so again we are faced with the problem of making the right choice.

Lets focus on the generator M^{-I} we might guess:

$$M^{-I} = x_0^- p^I - x_0^I p^- - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^-\alpha_n^I - \alpha_{-n}^I\alpha_n^-\right) \tag{85}$$

A satisfactory quantum Lorentz generator should be both normal-ordered and hermitian. Every term is already normal-ordered, however the second term in the guess above is not hermitian, we can fix this by making the replacement $x_0^I p^- \longrightarrow \frac{1}{2}\left(x_0^I p^- + p^- x_0^I\right)$. Now we write p^- and the minus oscillators in terms of the Virasoro operators $L_0^\perp$ to obtain:

$$\begin{aligned} M^{-I} = x_0^- p^I &- \frac{1}{4\alpha' p^+}\left(x_0^I\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^I\right) \\ &- \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}\left(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp\right) \end{aligned} \tag{86}$$

The algebra of the Lorentz group requires $\left[M^{-I}, M^{-J}\right] = 0$, can we accomodate this result? Well lets see...

$$\begin{aligned} \left[M^{-I}, M^{-J}\right] = -\frac{1}{\alpha' {p^+}^2}\sum_{m=1}^{\infty}&\left(\alpha_{-m}^I \alpha_m^J - \alpha_{-m}^J \alpha_m^I\right) \times \\ &\left\{m\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{m}\left[\frac{1}{24}(D-2) + a\right]\right\} \end{aligned} \tag{87}$$

So it must be:

$$m\left[1 - \frac{1}{24}(D-2)\right] + \frac{1}{m}\left[\frac{1}{24}(D-2) + a\right] = 0 \qquad \forall m \in \mathbb{Z}^+ \tag{88}$$

This is only possible if each bracket separately vanishes, thus we conclude $D = 26$ and $a = -1$. So imposing that the algebra of the Lorentz group be satisfied we have found the dimension of the spacetime and also the ordering constant of the transverse Virasoro mode. Indeed the argument $\sum_{p=1}^{\infty} p = -\frac{1}{12}$ provided the same result for a but did not give any clue about the dimension of the spacetime.

5.5 State space

In the quantum theory the mass spectrum is discrete and there are also massless states with different polarizations so the quantum theory is able to account for photons.

The ground states are specified by p^+ and $\vec{p}_T$.

To create states from the ground states we act on them with creation operators. The general basis state $|\lambda\rangle$ of the state space can be written as:

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_n^{I\dagger}\right)^{\lambda_{n,I}}|p^+, \vec{p}_T\rangle \tag{89}$$

We can define the operator $N^{\perp} \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I$. It is called number operator since when it acts on a basis state it returns as eigenvalue the sum of the mode numbers of the creation operators appearing in the state.

It satisfies the following commutation relations:

$$\left[N^{\perp}, a_n^{I\dagger}\right] = n a_n^{I\dagger} \tag{90}$$

$$\left[N^{\perp}, a_n^{I}\right] = -n a_n^{I} \tag{91}$$

The mass operator is in terms of the number operator:

$$M^2 = \frac{1}{\alpha'}\left(-1 + N^{\perp}\right) \tag{92}$$

6 Relativistic quantum closed strings

6.1 Mode expansion and commutation relations

For closed strings the gauge fixing is:

$$n \cdot X = \alpha'(n \cdot p)\tau \qquad n \cdot p = 2\pi n \cdot \mathcal{P}^{\tau} \tag{93}$$

The parameterization is chosen so that $\sigma \in [0, 2\pi]$. We showed that $n \cdot \mathcal{P}^{\sigma}$ is constant along the string, furthermore for open strings this quantity vanishes since it vanish at the endpoints. We commented at the end of section 4.1 that these condition could be imposed even for closed strings.

The general solution for the wave equation is:

$$X^{\mu}(\tau, \sigma) = X_L^{\mu}(\tau + \sigma) + X_R^{\mu}(\tau - \sigma) \tag{94}$$

The points $\sigma = 0$ and $\sigma = 2\pi$ are to be identified since the parameter space of a closed string is a cylinder and so we have a periodicity condition rather than a boundary condition:

$$X^{\mu}(\tau, \sigma) = X^{\mu}(\tau, \sigma + 2\pi), \qquad \forall \tau, \sigma \tag{95}$$

Setting $u \equiv \tau + \sigma$ and $v \equiv \tau - \sigma$ we can write:

$$X_L^{\mu}(u + 2\pi) - X_L^{\mu}(u) = X_R^{\mu}(v) - X_R^{\mu}(v - 2\pi) \tag{96}$$

So we can write the mode expansion:

$$X_L^{\mu'}(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \bar{\alpha}_n^{\mu} e^{-inu} \tag{97}$$

$$X_R^{\mu'}(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-inv} \tag{98}$$

Integrating these relations we get:

$$X_L^\mu(u) = \frac{1}{2} x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}} \bar{\alpha}_0^\mu u + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-inu} \tag{99}$$

$$X_R^\mu(v) = \frac{1}{2} x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu v + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-inv} \tag{100}$$

Upon integration two zero modes have appeared $x_0^{L\mu}$ and $x_0^{R\mu}$ but only the sum of them is relevant. If the space is not simply connected then both modes will play a role. Condition (96) implies $\bar{\alpha}_0^\mu = \alpha_0^\mu$ this implies that quantum closed string theory has only one momentum operator.

Assembling the left and right movers we obtain:

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(x_0^{L\mu} + x_0^{R\mu}\right) + \sqrt{2\alpha'}\alpha_0^\mu \tau + i \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right) \tag{101}$$

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right) \tag{102}$$

We find for the momentum:

$$p^\mu = \int_0^{2\pi} \mathcal{P}^{\tau\mu}(\tau,\sigma) d\sigma = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \sqrt{2\alpha'}\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu \tag{103}$$

To quantize the theory we impose equal time commutation relations:

$$\left[x_0^-, p^+\right] = -i \tag{104}$$

$$\left[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = i\eta^{IJ}\delta(\sigma - \sigma') \tag{105}$$

Is more useful to cast the commutation relations in a set of discrete commutation relations among oscillators. We find:

$$\left[\bar{\alpha}_m^I, \bar{\alpha}_n^J\right] = m\delta_{m+n,0}\eta^{IJ} \tag{106}$$

$$\left[\alpha_m^I, \alpha_n^J\right] = m\delta_{m+n,0}\eta^{IJ} \tag{107}$$

Closed string theory thus has the operator content of two copies of open string theory, except for zero modes. The momentum zero modes are equal ($\alpha_0^I = \bar{\alpha}_0^I$), and there is only one set of coordinates zero modes x_0^I,x_0^-.

6.2 Closed string Virasoro operators

Having two set of oscillators we also have two sets of transverse Virasoro modes:

$$\left(\dot{X}^I + X^{I'}\right)^2 = 4\alpha' \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}\sum_{p\in\mathbb{Z}} \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I\right) e^{-in(\tau+\sigma)} \equiv 4\alpha' \sum_{n\in\mathbb{Z}} \bar{L}_n^\perp e^{-in(\tau+\sigma)} \tag{108}$$

$$\left(\dot{X}^I - X^{I'}\right)^2 = 4\alpha' \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_p^I \alpha_{n-p}^I\right) e^{-in(\tau-\sigma)} \equiv 4\alpha' \sum_{n\in\mathbb{Z}} L_n^\perp e^{-in(\tau-\sigma)} \tag{109}$$

$$\bar{L}_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I \tag{110}$$

$$L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_p^I \alpha_{n-p}^I \tag{111}$$

Plugging the definitions in (54) we obtain:

$$\dot{X}^- + X^{-'} = \frac{2}{p^+}\sum_{n\in\mathbb{Z}} \bar{L}_n^\perp e^{-in(\tau+\sigma)} \tag{112}$$

$$\dot{X}^- - X^{-'} = \frac{2}{p^+}\sum_{n\in\mathbb{Z}} L_n^\perp e^{-in(\tau-\sigma)} \tag{113}$$

But from mode expansion (102) we also have:

$$\dot{X}^- + X^{-'} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^- e^{-in(\tau+\sigma)} \tag{114}$$

$$\dot{X}^- - X^{-'} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^- e^{-in(\tau-\sigma)} \tag{115}$$

We compare the equations above to read off the expressions for the minus oscillators:

$$\sqrt{2\alpha'}\bar{\alpha}_n^- = \frac{2}{p^+}\bar{L}_n^\perp \tag{116}$$

$$\sqrt{2\alpha'}\alpha_n^- = \frac{2}{p^+}L_n^\perp \tag{117}$$

Since $\alpha_0^- = \bar{\alpha}_0^- \Longrightarrow L_0^\perp = \bar{L}_0^\perp$.

To fix the ordering ambiguities in the operators $L_0^\perp$ and $\bar{L}_0^\perp$ we define them to be ordered operators without any additional constansts.

$$\bar{L}_0^\perp = \frac{\alpha'}{4} p^I p^I + \bar{N}^\perp \tag{118}$$

$$L_0^\perp = \frac{\alpha'}{4} p^I p^I + N^\perp \tag{119}$$

In closed string theory the critical dimension of spacetime is dictated as well by the requirement that the quantum theory is Lorentz invariant. The same result $D = 26$ is found. It means that both kinds of strings open and closed can coexist.

The ambiguities due to the ordering issues in $\bar{L}_0^\perp$ and $L_0^\perp$ are also fixed by the condition of Lorentz invariance, just as it happened for open strings. Equations (116) and (117) for $n = 0$ and once the ordering ambiguities have been taken into account become:

$$\sqrt{2\alpha'}\bar{\alpha}_0^- = \frac{2}{p^+}\left(\bar{L}_0^\perp - 1\right) \tag{120}$$

$$\sqrt{2\alpha'}\alpha_0^- = \frac{2}{p^+}\left(L_0^\perp - 1\right) \tag{121}$$

In account of these two equations we have:

$$\sqrt{2\alpha'}\alpha_0^- \equiv \frac{1}{p^+}\left(L_0^\perp + \bar{L}_0^\perp - 2\right) = \alpha' p^- \tag{122}$$

Having an expression for p^- we can write down the mass operator:

$$M^2 = \frac{2}{\alpha'}\left(N^\perp + \bar{N}^\perp - 2\right) \tag{123}$$

7 D-branes

In section 5.1 we quantized open strings in a space-filling D-brane. Now we go on to tackle more general cases in order of increasing complexity.

7.1 Single Dp-brane

When we have a Dp-brane we can set up spacetime coorinates x^μ in such a way that each coordinate is either tangential or normal to the brane.

$$
\begin{aligned}
&x^0, x^1, ..., x^p \qquad Dp \quad tangential \quad coordinates \\
&x^{p+1}, x^{p+2}, ..., x^d \qquad Dp \quad normal \quad coordinates
\end{aligned}
\tag{124}
$$

The string coordinates are equally split:

$$
\begin{aligned}
&X^0, X^1, ..., X^p \qquad Dp \quad tangential \quad coordinates \\
&X^{p+1}, X^{p+2}, ..., X^d \qquad Dp \quad normal \quad coordinates
\end{aligned}
\tag{125}
$$

The endpoints of the open string must end on the Dp-brane, therefore the string coordinates normal to the brane must satisfy Dirichlet boundary conditions:

$$
X^a(\tau, \sigma) \mid_{\sigma=0} = X^a(\tau, \sigma) \mid_{\sigma=\pi} = \bar{x}^a, \qquad a = p+1, ..., d \tag{126}
$$

Where $\bar{x}^a$ are a set of constants that specify the location of the brane. The X^a coordinates are called DD coordinates since both endpoints satisfy a Dirichlet boundary condition. The open string endpoints can move freely along the directions tangential to the D-brane. As a result, the string coordinates tangential to the D-brane satisfy Neumann boundary conditions:

$$
X^{m'}(\tau, \sigma)|_{\sigma=0} = X^{m'}(\tau, \sigma)|_{\sigma=\pi} = 0 \qquad m = 0, 1, ..., p \tag{127}
$$

These are called NN coordinates since both endpoints satisfy a Neumann boundary condition.

In order to use the light-cone gauge we need at least one spatial NN coordinate that can be used to define $X^{\pm}$. We need to assume $p \geq 1$ and so our analysis does not apply to D0-branes, this does not mean that D0-branes do not exists but is rather a failure of our approach. In order to describe D0-branes we need to use a Lorentz covariant quantization (Polyakov).

The NN coordinates $X^i(\tau, \sigma)$ satisfy exactly the same conditions that are satisfied by the light-cone $X^I(\tau, \sigma)$ of open strings attached to a D25-brane. The coordinate $X^+ = \beta\alpha' p^+ \tau$ by our gauge fixing, X^- was determined in terms of the transverse light-cone coordinates, the only difference is that in the presence of a Dp-brane the index I is splitted into two indices i, a depending on whether the transverse coordinate is tangential or normal to the D-brane. Therefore:

$$
\begin{aligned}
\dot{X}^- \pm X^{-'} &= \frac{1}{2\alpha'} \frac{1}{2p^+} \left(\dot{X}^I \pm X^{I'} \right)^2 = \\
&= \frac{1}{2\alpha'} \frac{1}{2p^+} \left(\left(\dot{X}^i \pm X^{i'} \right)^2 + \left(\dot{X}^a \pm X^{a'} \right)^2 \right)
\end{aligned}
\tag{128}
$$

In the presence of a Dp-brane the Lorentz symmetry of spacetime breaks:

$$SO(1,d) \Longrightarrow SO(1,p) \times SO(d-p) \tag{129}$$

The mode expansion for the X^i coordinates is known so we focus on the X^a coordinates. They are solutions of the wave equation so:

$$X^a(\tau,\sigma) = \frac{1}{2}\left(f^a(\tau+\sigma) + g^a(\tau-\sigma)\right) \tag{130}$$

The Dirichlet boundary condition at $\sigma = 0$ dictates:

$$X^a(\tau,0) = \frac{1}{2}\left(f^a(\tau+\sigma) + g^a(\tau-\sigma)\right) = \bar{x}^a \tag{131}$$

So then:

$$X^a(\tau,\sigma) = \bar{x}^a + \frac{1}{2}\left(f^a(\tau+\sigma) - f^a(\tau-\sigma)\right) \tag{132}$$

The boundary condition at $\sigma = \pi$ reads:

$$f^a(\tau+\pi) = f^a(\tau-\pi) \tag{133}$$

So being $f^a(u)$ a 2π periodic function it admits a Fourier series:

$$f^a(u) = \tilde{f}^a_0 + \sum_{n=1}^{\infty}\left(\tilde{f}^a_n \cos(nu) + \tilde{g}^a_n \sin(nu)\right) \tag{134}$$

Or in terms of τ and σ:

$$X^a(\tau,\sigma) = \bar{x}^a + \sum_{n=1}^{\infty}\left(-\tilde{f}^a_n \sin(n\tau)\sin(n\sigma) + \tilde{g}^a_n \cos(n\tau)\sin(n\sigma)\right) \tag{135}$$

We can rename the expansion coeffcients to write the mode expasion in terms of oscillators:

$$X^a(\tau,\sigma) = \bar{x}^a + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^a_n e^{-in\tau}\sin(n\sigma) \tag{136}$$

Note that in the expansion of the normal coordinates there is no linear term in τ and so the string does not have net time-averaged momentum in the normal directions. This certainly makes sense since strings must remain attached to the D-brane.

Quantization process is as always, we need to establish commutation relations among coordinates and conjugate momenta:

$$\left[X^a(\tau,\sigma), \dot{X}^b(\tau,\sigma')\right] = 2\pi\alpha' i\delta^{ab}\delta(\sigma-\sigma') \tag{137}$$

In terms of oscillators we obtain just as before:

$$\left[\alpha_m^a, \alpha_n^b\right] = m\delta^{ab}\delta_{m+n,0}, \qquad m, n \neq 0 \tag{138}$$

The mass operator is:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\left[\alpha_{-n}^i\alpha_n^i + \alpha_{-n}^a\alpha_n^a\right] - 1\right) \tag{139}$$

A generic basis state is:

$$\left[\prod_{n=1}^{\infty}\prod_{i=2}^{p}\left(a_n^{i\dagger}\right)^{\lambda_{n,i}}\right]\left[\prod_{m=1}^{\infty}\prod_{a=p+1}^{d}\left(a_m^{a\dagger}\right)^{\lambda_{m,a}}\right]|p^+, \vec{p}\rangle \tag{140}$$

Lets study the mass and the transformation behaviour of some states.

- The ground state:

$$|p^+, \vec{p}\rangle\,, \qquad M^2 = -\frac{1}{\alpha'} \tag{141}$$

This is just a Lorentz scalar field on the brane.

- First excited states, tangent to the brane:

$$a_1^{i\dagger}|p^+, \vec{p}\rangle\,, \qquad i = 2, ..., p \qquad M^2 = 0 \tag{142}$$

These are $(p-1)$ massless states. They carry an index that lives on the brane where there is $SO(1,p)$ Lorentz symmetry so they transform as a Lorentz vector on the brane. Since the number of states is equal the spacetime dimensionality of the brane minus 2, these are clearly photon states. We have found that a Dp-brane has a Maxwell field living on its world-volume.

- First excited states, normal to the brane:

$$a_1^{a\dagger}|p^+, \vec{p}\rangle\,, \qquad a = p+1, ..., d \qquad M^2 = 0 \tag{143}$$

There are $(d-p)$ massless states living on the brane. The index a is not a Lorentz index so these are just scalar fields. We have found that on a Dp-brane there is a massless scalar for every normal direction.

7.2 Open strings between parallel Dp-branes

The setting in this case is two parallel D-branes of the same dimensionality. The first brane is located at $x^a = \bar{x}_1^a$ and the second at $x^a = \bar{x}_2^a$, in this situation we can have strings that begin and end at the same brane or strings that stretch from one brane to the other. We talk about different sectors, these are labeled by the Chan-Paton indices. If there are two branes there are four different sectors: $[11], [22], [12]$ and $[21]$.

Lets study the sector [12] since is the only novelty. The NN string coordinates X^+, X^- and X^i are quantized just as before. The mode expansion for stretched strings will be different since the boundary conditions have changed. We find:

$$X^a(\tau, \sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\,\frac{\sigma}{\pi} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\sin(n\sigma) \tag{144}$$

Again we find no linear term in τ and so no time-averaged momentum in the x^a direction. We set $\sqrt{2\alpha'}\alpha_0^a = \frac{1}{\pi}\left(\bar{x}_2^a - \bar{x}_1^a\right)$, there is no contradiction because the interpretation of α_0 as momentum requires that it appears in $\dot{X}$.

The mass operator picks up an extra term:

$$M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}\left(N^\perp - 1\right) \tag{145}$$

- Ground state:

$$|p^+, \vec{p}; [12]\rangle, \qquad M^2 = -\frac{1}{\alpha'} + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 \tag{146}$$

The mass can be negative, zero or positive depending on the separation between the branes. If the separation of the branes vanishes we find tachyonic states, for the critical separation $|\bar{x}_2^a - \bar{x}_1^a| = 2\pi\sqrt{\alpha'}$ the ground state represents a massless scalar field and for larger separations a massive scalar field.

- Normal first excited states:

$$a_1^{a\dagger}\,|p^+, \vec{p}; [12]\rangle, \qquad a = p+1, ..., d, \qquad M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 \tag{147}$$

These are $(d-p)$ massive states. Since there is no Lorentz symmetry outside the brane these are just scalar fields.

- Tangent first excited states:

$$a_1^{i\dagger}\,|p^+, \vec{p}; [12]\rangle\,, \qquad i = 2, ..., p, \qquad M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 \tag{148}$$

These are $(p-1)$ massive states. They carry an index corresponding to the $(p+1)$-dimensional spacetime, the world-volume of the brane. We might think that these states make up a massive Maxwell gauge field, but this is not possible we are missing one state.

A massive gauge field has one more state than a massless one. In a D-dimensional spacetime, a massless gauge field has $D-2$ states, while a massive gauge field has $D-1$ states. In the present case one of the tachyons states must join the $(p-1)$ states to form the massive vector. At the end we are left with a massive vector and $d-p-1$ massive scalars.

It turns out that the state that joins in to make up the massive vector is:

$$\sum_a (\bar{x}_2^a - \bar{x}_1^a)\, a_1^{a\dagger}\,|p^+, \vec{p}; [12]\rangle \tag{149}$$

In the limit as the two branes approach each other they are still distinguishable and we still have the four sectors. The massless open string states which represent strings extending from brane one to brane two include a massless gauge field and $(d-p)$ massless scalars. This is the same field contents as that of a sector where strings begin and end on the same D-brane. When the two D-branes coincide we therefore get a total of four massless gauge fields. These gauge fields actually interact with one another - in the string picture they do so by the process of joining endpoints. Theories of interacting gauge fields are called Yang-Mills theories. More precisely we have a $U(2)$ Yang-Mills theory with some additional interactions that become negligible at low energies. In general N coincident D-branes carry U(N) massless gauge fields. This result is quite important since our current understanding of high energy physics is based on Yang-Mills theories. Electroweak theory and QCD are examples of Yang-Mills theories.

8 T-duality of closed strings

String theory requires as a consistency condition that the spacetime dimension be a certain number. In section 5.4 we found that $D = 26$. It may be thought that such a strong requirement should be perceived just by our senses, however classical physics and even quantum field theory work in four dimensions so all the coordinates can not be on the same footing. String theory offers an explanation for the fact that at low energies many dimensions remain hidden. The idea is that these extra dimensions are compactified.

We will find that in the presence of compact dimensions a new symmetry arises.

8.1 Compact dimensions

Imagine a spacetime where one dimension, say x^{25}, has been compactified, this means: $x \sim x + 2\pi R$. We drop the superscript and refer to x^{25} as simply x. In this case the topology of spacetime is non-trivial in the sense that not every closed curve can be shrunk to a point, namely those that wind around the compact dimension. In this circumstances the periodicity condition does not apply, instead:

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + m(2\pi R) \tag{150}$$

Where m is the winding number. We define the winding as: $\omega \equiv \frac{mR}{\alpha'}$.

8.2 Left and right movers

The left and right movers are:

$$X_L(u) = \frac{1}{2}x_0^L + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0 u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}_n}{n}e^{-inu} \tag{151}$$

$$X_R(v) = \frac{1}{2}x_0^R + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0 v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-inv} \tag{152}$$

The condition (150) translates into:

$$\bar{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'}\omega \tag{153}$$

The momentum of the string along the compact dimension is:

$$p = \frac{1}{2\pi\alpha'}\int_0^{2\pi}\left(\dot{X}_L + \dot{X}_R\right)d\sigma = \frac{1}{\sqrt{2\alpha'}}\left(\alpha_0 + \bar{\alpha}_0\right) \tag{154}$$

So we have:

$$p = \frac{1}{\sqrt{2\alpha'}}\left(\alpha_0 + \bar{\alpha}_0\right) \tag{155}$$

$$\omega = \frac{1}{\sqrt{2\alpha'}}\left(\alpha_0 - \bar{\alpha}_0\right) \tag{156}$$

These expressions suggest that the winding ω is on the same footing as the momentum p. Since we have two momenta we should expect two coordinates zero modes. We rewrite $x_0^L = x_0 + q_0$ and $x_0^r = x_0 - q_0$, thus introducing the average coordinate x_0 and the coordinate difference q_0. We can then write:

$$X_L(\tau+\sigma) = \frac{1}{2}(x_0+q_0) + \frac{\alpha'}{2}(p+\omega)(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\bar{\alpha}_n}{n}e^{-in(\tau+\sigma)} \quad (157)$$

$$X_R(\tau-\sigma) = \frac{1}{2}(x_0-q_0) + \frac{\alpha'}{2}(p-\omega)(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in(\tau-\sigma)} \quad (158)$$

The full coordinate is:

$$X(\tau,\sigma) = x_0 + \alpha' p\tau + \alpha'\omega\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\bar{\alpha}_n e^{-in\sigma} + \alpha_n e^{in\sigma}\right) \quad (159)$$

In this expansion the only evidence of a compact dimension is the winding term $\alpha'\omega\sigma$. The zero mode q_0 is not present here.

8.3 Quantization and commutators relations

The commutation relation $[X(\tau,\sigma), \mathcal{P}(\tau,\sigma')] = i\delta(\sigma-\sigma')$ leads to:

$$[\bar{\alpha}_m, \bar{\alpha}_n] = [\alpha_m, \alpha_n] = m\delta_{m+n,0}, \qquad [\alpha_m, \bar{\alpha}_n] = 0 \quad (160)$$

On account on (155),(156) and (160) it follows: $[p,\omega] = 0$
Also: $[p,\bar{\alpha}_n] = [p,\alpha_n] = [\omega,\bar{\alpha}_n] = [\omega,\alpha_n] = 0$
To find the commutation relations of x_0 with the other operators we use the result:

$$\left[X(\tau,\sigma), \left(\dot{X}(\tau,\sigma) \pm X'(\tau,\sigma')\right)\right] = 2\pi\alpha' i\delta(\sigma-\sigma') \quad (161)$$

Integrated over $\sigma \in [0,2\pi]$ we have:

$$\left[x_0, \left(\dot{X}(\tau,\sigma) \pm X'(\tau,\sigma')\right)\right] = \alpha' i \quad (162)$$

We thus learn: $[x_0,\alpha_0] = [x_0,\bar{\alpha}_0] = i\sqrt{\frac{\alpha'}{2}}$. And also: $[x_0,p] = i$ and $[x_0,\omega] = 0$.

Note that all the operators than appear in the mode expansion of $X(\tau,\sigma)$ commutes with the winding ω. We could thus think that the winding is a constant of the motion and this would mean that each winding corresponds to a different sector. This interpretation treats p and ω in quite different way. Another interpretation is that ω is an operator, just like p, and that the eigenvalues of ω correspond to the various possible windings. This will turn out to be the more natural interpretation.

The compactification of the x coordinate implies that after a translation by $2\pi R$ we return to the same point so since p is the generator of translations we have:

$$1 = e^{ip2\pi R} \Longrightarrow p = \frac{n}{R}, \qquad n \in \mathbb{Z} \tag{163}$$

Note that the result of the compactification has been to lose some momentum states, but also we gained some states the winding states. Furthermore note that the eigenvalues of momentum and winding transmute into one another under the transformation $R \Longrightarrow \frac{\alpha'}{R}$:

$$p = \frac{n}{R} \Longrightarrow \frac{nR}{\alpha'} \tag{164}$$

$$\omega = \frac{mR}{\alpha'} \Longrightarrow \frac{m}{R} \tag{165}$$

This feature is the responsible for T-duality. T-duality states that the physics is the same if the compactification radius is R or if it is $\tilde{R} = \frac{\alpha'}{R}$. To prove it in full detail we would have to find an isomorphism between the two pictures that preserves all the commutation relations and takes one Hamiltonian into the other.

9 T-duality of open strings

Lets consider the propagation of an open string in a space-filling D25-brane, in this situation all coordinates are NN coordinates and the endpoints can move around the whole space. Assume that one dimension has been compactified: $x^{25} \sim x^{25} + 2\pi R$. In the presence of compact dimensions closed strings exhibit fundamentally new states: non-trivial winding states are now possible. However open strings exhibit no fundamentally new states in the presence of a compact dimension, an open string can always be shrunk to a point and so there is no winding ω^{25}. The open string momentum in the compactified dimension is quatized: $p^{25} = \frac{n}{R}$.

Now consider the dual picture, wich is just as before but this time the compactification radius is the dual radius $\tilde{R} = \frac{\alpha'}{R}$. In this new spacetime the momentum is quantized with eigenvalues $p^{25} = \frac{nR}{\alpha'}$. It is clear that the spectrum of these two pictures is not the same. In seems that T-duality is not a symmetry in the presence of open strings.

There is a solution out of the problem, the key is to allow that T-duality does not only change the compactification radius but also the dimension of the D-brane. In the dual picture the radius of compactification is $\tilde{R} = \frac{\alpha'}{R}$ and the brane is a D24-brane, in this situation the X^{25} coordinate is a DD coordinate. By convention we set $x^{25} = 0$ to be the position of the brane along the compact dimension.

In this dual world all open string endpoints must remain attached to the D24-brane defined by $x^{25} = 0$. As a result, there are new open string configurations that can not be contracted away. Since X^{25} is now a DD coordinate there is no momentum along this direction. This is remarkable, momentum has disappeared but instead winding has appeared. This mechanism preserves T-duality even in the presence of open strings.

Lets show how this ideas work explicitly. Recall the mode expansion of a NN coordinate. For $X^{25}(\tau,\sigma) \equiv X(\tau,\sigma)$:

$$X(\tau,\sigma) = x_0 + \sqrt{2\alpha'}\alpha_0\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n\cos(n\sigma)e^{-in\tau} \tag{166}$$

We also have: $\alpha_0 = \sqrt{2\alpha'}p = \sqrt{2\alpha'}\frac{n}{R}$ since the momentum on the circle is quantized.

We can separate the string coordinate X into left and right movers:

$$X(\tau,\sigma) = X_L(\tau,\sigma) + X_R(\tau-\sigma) \tag{167}$$

$$X_L = \frac{1}{2}(x_0 + q_0) + \sqrt{\frac{\alpha'}{2}}\alpha_0(\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}e^{-in\sigma} \tag{168}$$

$$X_R = \frac{1}{2}(x_0 - q_0) + \sqrt{\frac{\alpha'}{2}}\alpha_0(\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}e^{in\sigma} \tag{169}$$

The constant q_0 is arbitrary. Inspired by closed string T-duality, where we reversed the sign of the right movers we define:

$$\tilde{X}(\tau,\sigma) \equiv X_L - X_R \tag{170}$$

$$\tilde{X}(\tau,\sigma) = q_0 + \sqrt{2\alpha'}\alpha_0\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau}\sin(n\sigma) \tag{171}$$

This is, in fact, the mode expansion for a string that stretches from one D-brane to another, as we can see by comapring with (144).

Our previous work with X^a and $\mathcal{P}^a$ shows that $\tilde{X}$ and $\tilde{\mathcal{P}}$ satisfy the canonical commutation relations. Therefore the duality transformation does not alter the commutation relations.

We can now turn to the physical interpretation. The coordinate $\tilde{X}$ is of DD type, since the endpoints are fixed: $\partial_\tau \tilde{X} = 0$ for $\sigma = 0$ and $\sigma = \pi$. When σ goes from 0 to π, the open string stretches an interval:

$$\tilde{X}(\tau, \pi) - \tilde{X}(\tau, 0) = \sqrt{2\alpha'}(\pi - 0) = 2\pi\alpha' p = 2\pi \frac{\alpha'}{R} n = 2\pi \tilde{R} n \tag{172}$$

This equations holds for every integer so the picture that emerges is that of an infinite collection of D24-branes with a uniform spacing $2\pi\tilde{R}$ along the x^{25} direction. Such a configuration is indeed equivalent to a single D24-brane at some fixed position on a circle of radius $\tilde{R}$.

It is interesting to note that T-duality interchanges boundary conditions. We have:

$$\partial_\sigma = X_L'(\tau + \sigma) - X_R'(\tau - \sigma) = \partial_\tau \tilde{X} \tag{173}$$

$$\partial_\tau = X_L'(\tau + \sigma) + X_R'(\tau - \sigma) = \partial_\sigma \tilde{X} \tag{174}$$

10 Electromagnetic fields on D-branes

In section 7.1, we learned that there is a Maxwell field living in the world volume of a Dp-brane, this Maxwell field couples to the open string endpoints in the following way:

$$S = \int d\tau d\sigma \mathcal{L}_{NG}(\dot{X}, X') + \int d\tau A_m(X) \frac{dX^m}{d\tau}|_{\sigma=\pi} - \int d\tau A_m(X) \frac{dX^m}{d\tau}|_{\sigma=0} \tag{175}$$

We will consider only constant background fields, in this situation the potential can be written as: $A_n(x) = \frac{1}{2} F_{mn} x^m$.

Varying the action we obtain the following boundary condition:

$$\mathcal{P}_m + F_{mn}\partial_\tau X^n = 0, \qquad \sigma = 0, \pi \tag{176}$$

In the light-cone gauge this equation simplyfies to:

$$\partial_\sigma X_m - 2\pi\alpha' F_{mn}\partial_\tau X^n = 0, \qquad \sigma = 0, \pi \tag{177}$$

Note that this boundary condition is of mixed type, it is neither Neumann nor Dirichlet.

10.1 D-branes with electric fields

Assume we have a Dp-brane with a background constant electric field that points along a compact direction. Our purpose is to describe how the situtation looks like in the dual picture. The claim is that we have a D(p-1)-brane moving along the dual circle and no electric field. The constraint that the brane can not move faster than the speed of light will constraint the magnitude of the electric field.

Consider a Dp-brane that wraps around a compact dimension x^{25} of radius R, and assume that the brane carries an electric field along this direction: $F_{25,0} = E_{25} \equiv E$. The boundary conditions are:

$$\partial_\sigma X^0 - \mathcal{E}\partial_\tau X = 0 \tag{178}$$

$$\partial_\sigma X - \mathcal{E}\partial_\tau X^0 = 0 \tag{179}$$

Where we have introduced the dimensionless electric field $\mathcal{E} = 2\pi\alpha' E$.

We can rewrite these boundary conditions in a more appropiate form using the light cone derivatives ∂_+ and ∂_-:

$$\partial_+ X^0 - \mathcal{E}\partial_+ X = \partial_- X^0 + \mathcal{E}\partial_- X \tag{180}$$

$$-\mathcal{E}\partial_+ X^0 + \partial_+ X = \mathcal{E}\partial_- X^0 + \partial_- X \tag{181}$$

Solving for $\partial_+ X^0$ and $\partial_+ X$ gives:

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \begin{pmatrix} \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2} \\ \frac{2\mathcal{E}}{1-\mathcal{E}} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix} \tag{182}$$

Indeed the ligh-cone derivatives have proven to be more useful in this case, since now the boundary conditions are expressed as an invertible linear relation. We also need to express in these coordinates the Dirichlet and Neumann boundary conditions, an the T-duality relations as well.

If X is a Neumann coordinate then:

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix} \tag{183}$$

If $\tilde{X}$ is a Dirichlet coordinate then:

$$\partial_+ \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} \tag{184}$$

The dual coordinate $\tilde{X}$ is obtained by changing the sign of the right movers in X, so the duality relations are:

$$\partial_+ X = \partial_+ \tilde{X} \tag{185}$$

$$\partial_- X = -\partial_- \tilde{X} \tag{186}$$

Now we can prove our claim. In the dual world there is a D(p-1)-brane that rotates along the dual radius and no electric field so we have that $\tilde{X}$ must be a Dirichlet coordinate, therefore:

$$\partial_+ \begin{pmatrix} X'^0 \\ \tilde{X}' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X'^0 \\ \tilde{X}' \end{pmatrix} \tag{187}$$

The prime mean that we are using coordinates where the D(p-1)-brane is at rest. We can boost it back to the frame where the original Dp-brane is at rest, the boost is:

$$\begin{pmatrix} X'^0 \\ \tilde{X}' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} \equiv M \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} \tag{188}$$

Substitue (188) into (187) to obtain the boundary conditions in the frame where the Dp-brane is at rest.

$$\partial_+ \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \partial_- \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} \tag{189}$$

Finally we can perform a T-duality transformation on the $\tilde{X}$ coordinate using (185) and (186), this can be accomplished by just introducing an appropiate matrix:

$$\begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} = M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ \tilde{X} \end{pmatrix} \tag{190}$$

$$\partial_+ \begin{pmatrix} X^0 \\ X \end{pmatrix} = \begin{pmatrix} \frac{1+\beta^2}{1-\beta^2} & \frac{2\beta}{1-\beta^2} \\ \frac{2\beta}{1-\beta} & \frac{1+\beta^2}{1-\beta^2} \end{pmatrix} \partial_- \begin{pmatrix} X^0 \\ X \end{pmatrix} \tag{191}$$

These are the open string boundary conditions for the theory dual to the moving D(p-1)-brane. As promised, they coincide with (182), which were written for a Dp-brane carrying an electric field, if we identify:

$$\mathcal{E} \equiv 2\pi\alpha' = \beta \tag{192}$$

The constrain $\beta \leq 1$ implies that $E_{MAX} = \frac{1}{2\pi\alpha'} = T_0$.

10.2 D-branes with magnetic fields

Assume we have a Dp-brane with a background constant magnetic field that points along a compact direction. Our purpose is to describe how the situtation looks like in the dual picture. The claim is that we have a D(p-1)-brane tilted along the dual circle and no magnetic field.

Consider a Dp-brane for which two directions of its world-volume lie on the $(x^2, \tilde{x}^3)$ plane. Assume that the $\tilde{x}^3$ dimension is compactified into a circle of radius $\tilde{R}$, and assume that the brane carries a magnetic field $F_{23} = B$. The boundary conditions are:

$$\partial_\sigma X^2 - \mathcal{B}\partial_\tau \tilde{X}^3 = 0 \tag{193}$$

$$\partial_\sigma \tilde{X}^3 + \mathcal{B}\partial_\tau X^2 = 0 \tag{194}$$

Where we have defined the dimensionless magnetic field $\mathcal{B} \equiv 2\pi\alpha' B$. In terms of light-cone derivatives the boundary conditions are:

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = \begin{pmatrix} \frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} & \frac{2\mathcal{B}}{1+\mathcal{B}^2} \\ \frac{-2\mathcal{B}}{1+\mathcal{B}^2} & \frac{1-\mathcal{B}^2}{1+\mathcal{B}^2} \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} \tag{195}$$

Since X'^2 and X'^3 are Neumann and Dirichlet respectively we can write:

$$\partial_+ \begin{pmatrix} X'^2 \\ X'^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X'^2 \\ X'^3 \end{pmatrix} \tag{196}$$

The prime indicates that this relations hold in the rotated frame, we can go back to the original fram by performing a rotation:

$$\begin{pmatrix} X'^2 \\ X'^3 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} \equiv R \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} \tag{197}$$

Plugging (197) in (196) we find:

$$\partial_+ \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} = R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R\partial_- \begin{pmatrix} X^2 \\ X^3 \end{pmatrix} \tag{198}$$

Now we perform the T-duality transformation that takes X^3 to $\tilde{X}^3$. This is done by including an additional matrix on the right hand side above:

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} \tag{199}$$

$$\partial_+ \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \partial_- \begin{pmatrix} X^2 \\ \tilde{X}^3 \end{pmatrix} \tag{200}$$

This expression matches (195) provided that:

$$\mathcal{B} \equiv 2\pi\alpha' B = -\tan\alpha \tag{201}$$

11 Nonlinear and Born-Infeld electrodynamics

In classical electrodynamics we are introduced to the concepts of electric and magnetic fields and we are encouraged to think about these as the agents that cause certain phenomena. When we study classical electrodynamics in the presence of materials the auxiliary fields $\vec{D}$ and $\vec{H}$ become more useful since their sources are the charges that we can modify at will in the laboratory but in the back of our mind we still think that the fundamental quantities are $\vec{E}$ and $\vec{B}$ together with all the charges: free and bound. The two sets $(\vec{E}, \vec{B})$ and $(\vec{D}, \vec{H})$ are related by some equations depending on the properties of the material at hand, usually this relations is highly nonlinear.

In nonlinear electrodynamics the vaccum itself becomes a very complex media that can be polarized. This suggest that we should think about $(\vec{D}, \vec{H})$ as being just as fundamental as $(\vec{E}, \vec{B})$ if not more. There are some Lagrangians that provide us with the nonlinear relations between the two sets in addition to the equation of motion for the set $(\vec{D}, \vec{H})$.

As we have seen in section 10.1 electric fields can not be arbitrarily large. Maxwell Lagrangian does not incorporate such a constraint, therefore we are motivated to seek new Lagrangians.

Consider the Lagrangian:

$$\mathcal{L} = -b^2 \sqrt{- \det \left(\eta_{\mu\nu} + \frac{1}{b} F_{\mu\nu} \right)} + b^2 \tag{202}$$

It has many nice features:

- It makes sense in any number of dimensions.
- It reduces to the Maxwell Lagrangian for small fields. Examine the Lagrangian for a spacetime of dimension four to obtain:

$$\mathcal{L} = -b^2 \sqrt{1 - \frac{2s}{b^2} - \frac{p^2}{b^4}} + b^2 \tag{203}$$

Where:

$$s \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(E^2 - B^2) = \mathcal{L}_{Maxwell} \tag{204}$$

$$p \equiv -\frac{1}{4} \left(\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \right) F_{\mu\nu} = \vec{E} \cdot \vec{B} \tag{205}$$

In case the is no magnetic field or more generally whenever p vanishes:

$$\mathcal{L} = -b^2 \sqrt{1 - \frac{2s}{b^2}} + b^2 = -b^2 \left(1 - \frac{s}{b^2} \right) + b^2 + \mathcal{O}(s^2) = s + \mathcal{O}(s^2) \tag{206}$$

- If there is no magnetic field the electric field is bounded. If $B = 0$:

$$\mathcal{L} = -b^2\sqrt{1 - \frac{E^2}{b^2}} + b^2 \tag{207}$$

Since the argument of the square root can not become negative we obtain: $E \leq b$, setting $b = 2\pi\alpha'$ we recover the result obtained in section 10.1.

- The self-energy of a point particle is finite.

We need to find the energy density:

$$\mathcal{L} = -b^2\sqrt{1 - \frac{E^2}{b^2}} + b^2 \tag{208}$$

$$\vec{D} = \frac{\partial \mathcal{L}}{\partial \vec{E}} = \frac{\vec{E}}{\sqrt{1 - \frac{E^2}{b^2}}} \Longrightarrow \vec{E} = \frac{\vec{D}}{\sqrt{1 + \frac{D^2}{b^2}}} \tag{209}$$

$$\mathcal{H} = \vec{E} \cdot \vec{D} - \mathcal{L} = b^2\sqrt{1 + \frac{D^2}{b^2}} - b^2 \tag{210}$$

Now we integrate over the whole space:

$$\begin{aligned} U_Q = \int d^3x \mathcal{H} &= b^2 \int_0^\infty 4\pi r^2 dr \left(\sqrt{1 + \left(\frac{Q}{4\pi b r^2}\right)^2} - 1\right) \\ &= \sqrt{\frac{b}{4\pi}} Q^{\frac{3}{2}} \int_0^\infty dx \left(\sqrt{1 + x^4} - x^2\right) \\ &= \frac{1}{4\pi}\frac{1}{3}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 b^{\frac{1}{2}} Q^{\frac{3}{2}} \end{aligned} \tag{211}$$

12 Bibliography

- Zwiebach, Barton. (2009) *A First Course in String Theory.*
- Polchinski, Joseph. (2005) *String Theory Vol 1.*
- Johnson, Clifford V. (2006) *D-Branes.*