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Classical String Theory

Proseminar in Theoretical Physics

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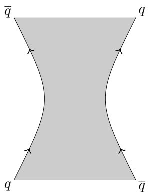
Outline

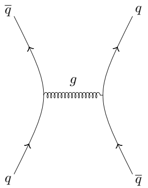
1. Introduction

1.1. Historical Overview

- 1969
  Nambu, Nielsen and Susskind propose a model for the interaction of quarks - quarks connected by 'strings'.
- 1970's
  Quantum Chromodynamics is recognized as the theory of strong interaction.
  String theory needs 10 dimensions to work, a very unlikely assumption (at this time) for an unneeded theory.

$\longrightarrow$ String theory went into the dustbin of history.





1. Introduction

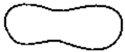
1.1. Historical Overview

- From all calculations (with closed strings) a massless particle with spin 2 appeares.
- This is exactly the property of the graviton
  $\rightarrow$ String theory always contains gravity.
- 1980's
  After the full discovery of the standard model, it was this fact which made string theory reappear as a promising candidate for a theory of quantum gravitation.

1. Introduction

1.2. The Theory

- In a nutshell:
  - String theorists propose **one dimensional fundamental objects** ('strings') instead of zero dimensional elementary particles.
  - The different elementary particles appear as different oscillation modes of strings.
  - Effects become important for small scale and high energy physics.
    $\rightarrow$ Planck length ($\sim 10^{-35} m$)
    $\rightarrow$ Planck energy ($\sim 10^{19} GeV$)



1. Introduction

1.2. The Theory

- Why String theory?
  - Promising (among other things due to the emergence of the graviton in a natural way) candidate for a theory of **quantum gravitation** or even a 'theory of everything'.
  - Brought many interesting and important **mathematical theories** to life.
  - Eludes the problem of **renormalization** in QFT.

Outline

2. The Relativistic String

2.1. Set Up

- Start by considering 'strings' as **fundamental objects** moving in a **(D dimensional)** Lorentzian spacetime (with Lorentzian metric $g_{\mu\nu}$) and obeying certain dynamical laws.
- Goal of this chapter:
  Find an action principle of the free bosonic string and study its dynamics.
- Idea
  Strings behave classically as the one dimensional analogon to point particles.
  $\rightarrow$ Review point particle action.

2. The Relativistic String

2.2. The Relativistic Point Particle

- Action should be a functional of the particle's path ('world line').
- Natural (and only) candidate: 'length' of the world line

$$S = -mc \int_\gamma \mathrm{d}s.$$

- Proportionality factor $-mc$ follows from classical limit ($v \to 0$).

2. The Relativistic String

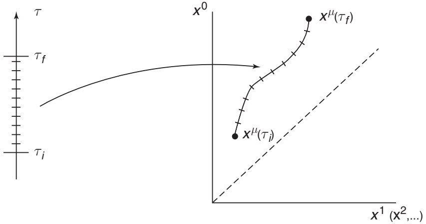
2.2. The Relativistic Point Particle

- parametrization $x^\mu(\tau)$.
- induced (1x1) metric on the parametrization domain (pullback of ambient metric $g_{\mu\nu}$)

$$\Gamma = \frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} g_{\mu\nu}$$

- volume (or length) form

$$\mathrm{d}s = \sqrt{|\Gamma|}\mathrm{d}\tau = \sqrt{-\dot{x}^\mu \dot{x}_\mu}\mathrm{d}\tau$$



$$x^\mu : [\tau_i, \tau_f] \longrightarrow \mathbb{R}^D$$

2. The Relativistic String

2.2. The Relativistic Point Particle

- Therefore, the **point particle action** is (writing $\dot{x}^\mu := \frac{dx^\mu}{d\tau}$)

$$\boxed{S_{\text{point particle}} = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\dot{x}^\mu \dot{x}_\mu}\, \mathrm{d}\tau}$$

- Better known form:
  In eigentime parametrization

$$x^\mu(\tau) = (c\,\tau, \mathbf{x}(\tau))$$

  the action reads

$$S = -mc^2 \int \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\, dt.$$

2. The Relativistic String

2.2. The Relativistic Point Particle

$$S = -mc \int \sqrt{-\dot{x}^\mu \dot{x}_\mu}\, \mathrm{d}\tau \qquad \Rightarrow \qquad \frac{d}{d\tau}\left(\frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}\right) = 0.$$

- Disadvantages
  - The **squareroot** is not easy to quantize.
  - The **massless case** is not covered.
  - The action has **primary constraints** (constraints not following from the equation of motion).
    From the definition $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$, it follows directly that $p^\mu p_\mu = -m^2 c^2$.
    $\longrightarrow$ More about primary constraints in report.

2. The Relativistic String

2.2. The Relativistic Point Particle

- Trick The Einbein Action:
  Define the action

$$S = \frac{1}{2} \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \, e(\tau) \left(e^{-2}(\tau) \left(\dot{x}^\mu(\tau) \right)^2 - m^2 \right)$$

  where $e(\tau)$ is an auxilliary function (which can be varied independently of x^μ).
- Varying with respect to x^μ and e gives

$$\begin{aligned} \frac{\delta S}{\delta e} &= 0 \qquad \Rightarrow \qquad \dot{x}^2 + e^2 m^2 &= 0 \\ \frac{\delta S}{\delta x^\mu} &= 0 \qquad \Rightarrow \qquad \frac{d}{d\tau} \left(e^{-1} \dot{x}^\mu \right) &= 0. \end{aligned}$$

2. The Relativistic String

2.2. The Relativistic Point Particle

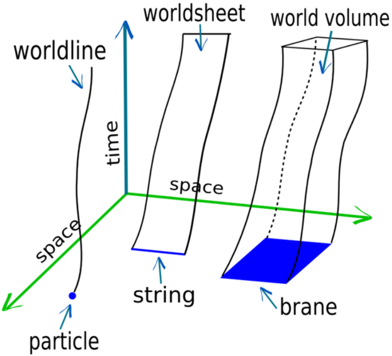
$$e = \frac{\sqrt{-\dot{x}^2}}{m} \ \& \ \frac{d}{d\tau}\left(e^{-1}\dot{x}^\mu\right) = 0 \quad \Rightarrow \quad \frac{d}{d\tau}\left(\frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}\right) = 0.$$

- The resulting equations of motion are **equivalent to the classical point particle** equation of motion together with the primary constraint.
- Found new equivalent action with
  - no squareroot
  - no primary constraint (old primary constraints are turned into equation of motion)

2. The Relativistic String

2.3. The General p-Brane Action

- Generalize: spatial 0-dim point $\rightarrow$ spatial (p-1)-dim objects in a D dimensional spacetime (called brane).
- represented by its p dimensional 'worldvolume' ($p = 1$ worldline, $p = 2$ worldsheet).



$$X^\mu : U \longrightarrow \mathbb{R}^D$$

2. The Relativistic String

2.3. The General p-Brane Action

- In **analogy to the point particle** the reparametrization invariant action of this world volume is

$$S \propto \text{Vol}(\text{world volume}).$$

- If $X^\mu(\tau^i)$ $(\mu \in \{1, ...D\}, i \in \{1, ..., p\})$ is a parametrization of the worldvolume of the brane, then the induced metric on the parametrization domain $\Gamma^{(p)}_{\alpha\beta}$ is the pullback of $g_{\mu\nu}$ under X^μ

$$\Gamma^{(p)}_{\alpha\beta} = \frac{\partial X^\mu}{\partial \tau^\alpha} \frac{\partial X^\nu}{\partial \tau^\beta} g_{\mu\nu}.$$

2. The Relativistic String

2.3. The General p-Brane Action

- The **general p-Brane action** therefore reads

$$\boxed{S_{\text{Brane}} = -T_p \int \sqrt{-\det\left(\frac{\partial X^\mu}{\partial \tau^\alpha}\frac{\partial X^\nu}{\partial \tau^\beta} g_{\mu\nu}\right)}\, \mathrm{d}^{p+1}\tau}$$

  where T_p is a proportionality factor.
- We are now ready to think about 'strings', or 1-branes.

2. The Relativistic String

2.4. The Nambu-Goto Action

- The **Nambu Goto action** is the action for a string or 1-brane:
  Parametrization
  $$(\tau, \sigma) \mapsto X^{\mu}(\tau, \sigma),$$
  then
  $$S_{\text{Nambu-Goto}} = -\frac{T_0}{c} \int \sqrt{-\det(\Gamma^{(1)}_{\alpha\beta})} d\tau d\sigma$$
  where the proportionality factor T_0 is called **string tension**.
- Calculating the determinant of $\quad (X'^{\mu} := \frac{\partial X^{\mu}}{\partial \sigma}, \dot{X}^{\mu} := \frac{\partial X^{\mu}}{\partial \tau})$

$$\Gamma^{(1)}_{\alpha\beta} = \frac{\partial X^{\mu}}{\partial \tau^{\alpha}} \frac{\partial X^{\nu}}{\partial \tau^{\beta}} g_{\mu\nu} = \begin{pmatrix} \dot{X} \cdot \dot{X} & \dot{X} \cdot X' \\ X' \cdot \dot{X} & X' \cdot X' \end{pmatrix}$$ gives

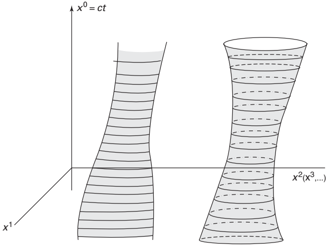
$$\boxed{S_{\text{NG}} = -\frac{T_0}{c} \int \int \sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2} d\tau d\sigma}$$

2. The Relativistic String

2.5. Open and Closed Strings

- For later: Consider two kinds of string, **open** and **closed**.
- Parametrization domain:

$$-\infty < \tau < \infty \qquad 0 < \sigma < \overline{\sigma}$$



- Closed Strings:
  $\rightarrow$ Worldsheet diffeo. to $\mathbb{R} \times S^1$,
  $\rightarrow$ periodicity conditions.
  $\rightarrow$ Convention $\overline{\sigma} = 2\pi$
- Open Strings:
  $\rightarrow$ Worldsheet diffeo. to $\mathbb{R} \times [0, \overline{\sigma}]$,
  $\rightarrow$ boundary conditions (later).
  $\rightarrow$ Convention $\overline{\sigma} = \pi$

2. The Relativistic String

2.6. The Polyakov Action

- In principle this is all, we have postulated the dynamics of the string and could start solving the Nambu Goto action.
- Problem:
  - Nambu-Goto has a square root
  - There are primary constraints (the so called Virasoro constraints $\rightarrow$ later)
- Trick:
  Find an action, which has no squareroot, no primary constraint and is equivalent to the NG action
  $\rightarrow$ same as einbein trick for the relativistic particle...

2. The Relativistic String

2.6. The Polyakov Action

- The **Polyakov action** is defined as

$$\boxed{S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\,\Gamma_{\alpha\beta}}$$

- $h_{\alpha\beta}$ is a symmetric (2x2) tensor, which can be **varied independently** of X^μ. It is taken as an intrinsic metric tensor on the parametrization domain with inverse $h^{\alpha\beta}$.
- $h_{\alpha\beta}$ is the two dimensional **analogon** to the einbein function e.
- The Polyakov action has **no primary constraints**.

2. The Relativistic String

2.7. The Equations of Motion

- To do: Find equations of motion and proof that they are equivalent to NG (+ primary constraints).
- Varying

$$\boxed{\delta S = -T \int d^2\sigma \left(\sqrt{-h} T_{\alpha\beta} \delta h^{\alpha\beta} + 2\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \partial_\beta X_\mu \right) \delta X^\mu \right) + \text{bnd term}}$$

where

- $T_{\alpha\beta}$ is the **energy momentum tensor**, the factor appearing when varying S with respect to $h^{\alpha\beta}$ or as a functional derivative

$$\begin{aligned} T_{\alpha\beta} = -\frac{1}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} &= \frac{1}{2} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \\ &= \frac{1}{2} \Gamma_{\alpha\beta} - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \Gamma_{\gamma\delta}. \end{aligned}$$

2. The Relativistic String

2.8. Equivalence of Polyakov and Nambu-Goto Action

- The **equations of motion** are

$$\boxed{\begin{aligned} T_{\alpha\beta} &= 0 \\ \partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right) &= 0 \end{aligned}}$$

- $\rightarrow$ see blackboard (equivalence of actions)
- Only for the two dimensional case the actions are equivalent.
- Remark:
  Only for minimizing $h^{\alpha\beta}$ fixed, we have $S_{\mathrm{NG}} = S_{\mathrm{P}}$ (in general not the same expression).

2. The Relativistic String

2.9. Symmetries of the Action

- Global Symmetries:
  - **Poincare Invariance**

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \qquad \text{where } a^\mu{}_\nu = -a_\nu{}^\mu$$

$$\delta h_{\alpha\beta} = 0$$

a general infinitesimal Poincare transformation.
$\rightarrow X^\mu$ transforms as expected as a vector, $h_{\alpha\beta}$ as a scalar.

2. The Relativistic String

2.9. Symmetries of the Action

- Local Symmetries:
  - **Reparametrization Invariance**
    Follows directly from the choice of action. Formally $\phi : \sigma^\alpha \mapsto \sigma'^\alpha$ a reparametrization of the parametrization domain. With $\xi^\alpha := \delta\phi^\alpha$

    $$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$

    and

    $$\delta h_{\alpha\beta} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + \partial_\alpha \xi^\gamma h_{\gamma\beta} + \partial_\beta \xi^\gamma h_{\alpha\gamma}.$$

  - **Weyl Symmetry**
    a bit more concealed symmetrie, invariance of the action wrt. Weyl scaling of the worldsheet metric h.

    $$\delta X^\mu = 0$$

    $$\delta h_{\alpha\beta} = 2\Lambda h_{\alpha\beta}$$

    where Λ is an arbitrary function.

Outline

3. Wave Equations and Solutions

3.1. Conformal Gauge and Weyl Scaling

- Goal of this chapter:
  Simplify equations of motion using the symmetries of the action and find solutions.
- Using **reparametrization invariance** we can simplify $h_{\alpha\beta}$.
- Claim:
  For any two dimensional Lorentzian (meaning signature (-1,1)) metric $h_{\alpha\beta}$ one can find coordinates σ^1, σ^2, such that

$$h_{\alpha\beta} = \Omega(\sigma^1, \sigma^2)\eta_{\alpha\beta}$$

  where $\eta_{\alpha\beta}$ is the Minkowski metric (and Ω a scalar function).

3. Wave Equations and Solutions

3.1. Conformal Gauge and Weyl Scaling

- Using **Weyl Scaling:**
  $\rightarrow$ gauge away Ω (set $\Omega = 1$).
- Having used all symmetries, we are in a gauge with

$$\boxed{h_{\alpha\beta} = \eta_{\alpha\beta}}$$

- Such a choice of coordinates is called a **conformal gauge**.
- We will from now on work in these coordinates (calling them τ and σ).

3. Wave Equations and Solutions

3.2. Equations of Motions and Boundary Conditions

- In conformal gauge, the Polyakov action takes a simple form

$$\boxed{S_{\mathrm{P}} = -\frac{T}{2}\int_0^{\overline{\sigma}} d\sigma \int_{\tau_i}^{\tau_f} d\tau\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{T}{2}\int_0^{\overline{\sigma}} d\sigma \int_{\tau_i}^{\tau_f} d\tau \left(\dot{X}^2 - X'^2\right)}$$

- The variations fulfill

$$\delta X^\mu(\sigma = 0, \overline{\sigma}) \text{ arbitrary for open strings}$$

$$\delta X^\mu(\sigma + 2\pi) = \delta X^\mu(\sigma) \text{ for closed strings}$$

$$\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$$

- Varying with respect to X^μ (last term vanishes for closed string):

$$\delta S_{\mathrm{P}} = T\int_0^{\overline{\sigma}} d\sigma \int_{\tau_i}^{\tau_f} d\tau\, \delta X^\mu \left(\partial_\sigma^2 - \partial_\tau^2\right) X_\mu - T\int_{\tau_i}^{\tau_f} d\tau\, \partial_\sigma X_\mu\, \delta X^\mu\big]_{\sigma=0}^{\sigma=\overline{\sigma}}$$

3. Wave Equations and Solutions

3.2. Equations of Motions and Boundary Conditions

- This results in the (wave-) **equation of motion and boundary conditions**

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$$

$$X^\mu(\sigma + 2\pi) = X^\mu(\sigma) \qquad \text{closed string}$$

$$X'^\mu(\sigma = 0, \pi) = 0 \qquad \text{open string.}$$

- **Neumann condition** for open string (=free ends). Dirichlet $(\delta X^\mu(\sigma = 0, \sigma = \overline{\sigma}) = 0$ would have worked too (fixed ends)).
- Should not forget to impose the **vanishing of the energy momentum tensor**

$$T_{\alpha\beta} = 0 \quad \Leftrightarrow \quad \left(\dot{X} \pm X'\right)^2 = 0 \quad \textbf{Virasoro constraint}$$

3. Wave Equations and Solutions

3.3. General Solution for the Closed String

- Equations of motion and boundary conditions closed string:

wave equation:	$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$
periodicity condition:	$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$
Virasoro constraints:	$\left(\dot{X} \pm X'\right)^2 = 0$

- In **light cone coordinates**

$$\sigma^\pm = \tau \pm \sigma \qquad \partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right)$$

the wave equation reads :

$$\partial_+ \partial_- X^\mu = 0$$

3. Wave Equations and Solutions

3.3. General Solution for the Closed String

- Solution to wave equation:

$$X^{\mu}(\sigma^{-},\sigma^{+}) = X_R^{\mu}(\sigma^{-}) + X_L^{\mu}(\sigma^{+})$$

- X_R and X_L are arbitrary functions only dependent on boundary conditions $\rightarrow$ **left - and right movers**.
- Closed String $\rightarrow$ Besides periodicity no boundary condition $\rightarrow$ X_R and X_L are **independent**.
- This is not the case for open strings $\rightarrow$ Neumann boundary condition connects them (open string = standing waves $\rightarrow$ reflected)

3. Wave Equations and Solutions

3.3. General Solution for the Closed String

- Claim (proof in report):
  The 2π periodicity of X^μ is equivalent to

$\partial_- X_R^\mu(\sigma^-)$ and $\partial_+ X_L^\mu(\sigma^+)$ are 2π periodic with the same zero-mode.

- Expand in **fourier series**:

$$\partial_- X_R^\mu(\sigma^-) = \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma^-}$$

$$\partial_+ X_L^\mu(\sigma^+) = \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{\infty} \overline{\alpha}_n^\mu e^{-in\sigma^+},$$

- Constants are choosen for convenience.
- α_n^μ and $\overline{\alpha}_n^\mu$ are generally **independent** (exception: $\alpha_0^\mu = \overline{\alpha}_0^\mu$).

3. Wave Equations and Solutions

3.3. General Solution for the Closed String

- Integrating these expressions $\rightarrow$ **oscillator expansion** (setting $p^\mu = \sqrt{4\pi T}\,\alpha_0^\mu = \sqrt{4\pi T}\,\overline{\alpha}_o^\mu$):

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^- + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}$$

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^+ + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\overline{\alpha}_n^\mu e^{-in\sigma^+}$$

- and together

$$\boxed{X^\mu(\sigma,\tau) = \underbrace{x^\mu + \frac{1}{2\pi T}p^\mu\tau}_{\text{center of mass motion}} + \underbrace{\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)e^{-in\tau}}_{\text{oscillation of the string}}}$$

3. Wave Equations and Solutions

3.3. General Solution for the Closed String

- Properties:
  - X^μ real implies

    $$x^\mu, p^\mu \text{ are both real} \qquad (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \qquad (\overline{\alpha}_n^\mu)^\dagger = \overline{\alpha}_{-n}^\mu.$$

  - x^μ is the **center of mass** of the string at $\tau = 0$:

    $$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, X^\mu(\sigma, 0) = x^\mu$$

  - The canonical τ-momentum is

    $$P_\tau^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = \frac{\partial}{\partial \dot{X}_\mu}\left(\frac{T}{2}\left(\dot{X}^2 - X'^2\right)\right) = T\dot{X}^\mu$$

    therefore the **total momentum** of the string is

    $$P_{c.o.m}^\mu = \int_0^{2\pi} \mathrm{d}\sigma\, P_\tau^\mu = p^\mu$$

3. Wave Equations and Solutions

3.4. General Solution for the Open String

- Equations of motion and boundary conditions open string:

wave equation: $\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$

boundary condition: $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$

Virasoro constraint: $\left(\dot{X} \pm X'\right)^2 = 0$

- Similar analysis leads to **oscillator expansion**:

$$\boxed{X^\mu(\tau,\sigma) = \underbrace{x^\mu + \frac{1}{\pi T} p^\mu \tau}_{\text{center of mass motion}} + \underbrace{\frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)}_{\text{oscillation of the string}}}$$

with x^μ, p^μ real and $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$.

- $\rightarrow$ Left and right movers are **not independent** anymore.

3. Wave Equations and Solutions

3.5. Virasoro Constraints

- We have found the general solution for the wave equation under consideration of boundary conditions.
- We still have to impose the **Virasoro constraints**

$$\left(\dot{X} \pm X'\right)^2 = 0 \qquad \text{or} \qquad T_{\alpha\beta} = 0.$$

- Where did they come from again?
  $\rightarrow$ primary constraints of NG action $\rightarrow$ equation of motion for $h^{\alpha\beta}$ in the P action $\rightarrow$ expressed as vanishing of the energy momentum tensor $T_{\alpha\beta}$ $\rightarrow$ equivalent to $\left(\dot{X} \pm X'\right)^2 = 0$.
- Can be seen as the string analogon to $p^\mu p_\mu = -m^2c^2$.

3. Wave Equations and Solutions

3.5. Virasoro Constraints

- Adviseable to discuss **Light cone coordinates** a bit further:
- The conformal metric looked like $h_{\alpha\beta} = \eta_{\alpha\beta}$. Therefore the **light cone metric** is $\eta_{++} = \eta{--} = 0$ and $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$.
- The **energy moment tensor** $T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_d X_\mu$ becomes in this coordinates with $h_{\alpha\beta} = \eta_\pm$

$$T_{++} = \frac{1}{2}\left(\partial_+ X\right)^2 \qquad T_{--} = \frac{1}{2}\left(\partial_- X\right)^2$$

$$T_{+-} = T_{-+} = 0$$

3. Wave Equations and Solutions

3.5. Virasoro Constraints

- What restriction impose the Virasoro constraints on α_n^μ, $\overline{\alpha}_n^\mu$?

$$0 \stackrel{!}{=} T_{--} = \frac{1}{2}(\partial_- X)^2 = \frac{1}{2}\left(\frac{1}{\sqrt{4\pi T}}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-in\sigma^-}\right)^2$$

$$= \frac{1}{8\pi T}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_n^\mu \alpha_{k\mu} e^{-i(n+k)\sigma^-} \underbrace{=}_{m=n+k} \frac{1}{8\pi T}\sum_n\sum_m \alpha_n \cdot \alpha_{m-n} e^{-im\sigma^-}$$

$$=: \frac{1}{4\pi T}\sum_m L_m e^{-im\sigma^-}$$ where $$\boxed{L_m := \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_n \cdot \alpha_{m-n}}$$

$$0 \stackrel{!}{=} T_{++} = \frac{1}{4\pi T}\sum_m \overline{L}_m e^{-im\sigma^+}$$ where $$\boxed{\overline{L}_m := \frac{1}{2}\sum_{n=-\infty}^{\infty}\overline{\alpha}_n \cdot \overline{\alpha}_{m-n}}$$

3. Wave Equations and Solutions

3.5. Virasoro Constraints

- Because $\sum_m L_m e^{-im\sigma^-} \stackrel{!}{=} 0$ for all σ^-, all so called **Virasoro modes** L_m must vanish.
- For the closed string one hast therefore to impose the **additional constraints on the modes**

$$\boxed{L_m = \overline{L}_m \stackrel{!}{=} 0}$$

- We have found the **most general solution** for the classical relativistic closed string:
  - $X^\mu(\sigma,\tau) = x^\mu + \frac{1}{2\pi T} p^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right) e^{-in\tau}$
  - with $L_m := \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_n \cdot \alpha_{m-n}$, $\overline{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \overline{\alpha}_n \cdot \overline{\alpha}_{m-n}$ fulfilling

$$L_m = \overline{L}_m = 0.$$

3. Wave Equations and Solutions

3.5. Virasoro Constraints

- For the open string the calculation is analog. Because X_R and X_L are not independent any more one has only one additional constraint

$$T_{++} = \frac{1}{4\pi T}\sum_m L_m e^{-im\sigma^+} \quad T_{--} = \frac{1}{4\pi T}\sum_m L_m e^{-im\sigma^-}$$

where

$$\boxed{L_m = \frac{1}{2}\sum_n \alpha_n \cdot \alpha_{m-n} \stackrel{!}{=} 0}$$

3. Wave Equations and Solutions

3.6. The Witt Algebra

- For later quantization: Algebraic considerations of the classical problem.
- The **Poisson bracket** for coordinate fields $X^\mu(\sigma)$, their conjugate momentum fields $P_\mu(\sigma)$ and functionals $f(X,P), g(X,P)$ is defined using the functional derivative

$$\{f,g\}_P = \int \mathrm{d}\sigma \, \frac{\delta f}{\delta X^\mu(\sigma)} \frac{\delta g}{\delta P_\mu(\sigma)} - \frac{\delta f}{\delta P_\mu(\sigma)} \frac{\delta g}{\delta X^\mu(\sigma)}$$

3. Wave Equations and Solutions

3.6. The Witt Algebra

- Interested in τ- propagation:

$X^\mu(\sigma,\tau)$ is seen as a field in σ

- **conjugate field momentum**

$$\Pi^\mu(\sigma,\tau) = \frac{\partial\mathcal{L}}{\partial\dot{X}_\mu}(\sigma,\tau) = T\dot{X}^\mu(\sigma,\tau)$$

- Then

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = 0 \quad \{\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\} = 0$$
$$\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\} = g^{\mu\nu}\delta(\sigma-\sigma')$$

$\rightarrow$ Next step: Calculate Hamiltonian

3. Wave Equations and Solutions

3.6. The Witt Algebra

- The τ Hamiltonian is

$$H = \int_0^{\overline{\sigma}} \mathrm{d}\sigma \, \dot{X}^\mu \Pi_\mu - \mathcal{L} = \frac{T}{2} \int_0^{\overline{\sigma}} \mathrm{d}\sigma \, \left(\dot{X}^2 + X'^2\right)$$

- With

$$\dot{X}^2 = (\partial_+ X + \partial_- X)^2 = 2T_{++} + 2T_{--} + 2\partial_+ X \cdot \partial_- X$$
$$X'^2 = (\partial_+ X - \partial_- X)^2 = 2T_{++} + 2T_{--} - 2\partial_+ X \cdot \partial_- X.$$

- Therefore the Hamiltonian (for τ propagation) is

$$H = 2T \int_0^{\overline{\sigma}} \mathrm{d}\sigma \, \left(T_{++} + T_{--}\right)$$

3. Wave Equations and Solutions

3.6. The Witt Algebra

- Closed String

$$T_{++} = \frac{1}{4\pi T} \sum_m \bar{L}_m e^{-im\sigma^+}$$

$$T_{--} = \frac{1}{4\pi T} \sum_m L_m e^{-im\sigma^-}$$

- Hamiltonian

$$\boxed{H = L_0 + \bar{L}_0}$$

- Open String

$$T_{++} = \frac{1}{4\pi T} \sum_m L_m e^{-im\sigma^+}$$

$$T_{--} = \frac{1}{4\pi T} \sum_m L_m e^{-im\sigma^-}$$

- Hamiltonian

$$\boxed{H = L_0}$$

- Goal: Calculate brackets of the Virasoro modes.

3. Wave Equations and Solutions

3.6. The Witt Algebra

- Closed String: From full solution

$$X^\mu(\sigma,\tau) = x^\mu + \frac{1}{2\pi T}p^\mu\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)e^{-in\tau}$$

$$\Pi^\mu(\sigma,\tau) = T\dot{X}^\mu = \frac{1}{2\pi}p^\mu + \sqrt{\frac{T}{4\pi}}\sum_{n\neq 0}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)e^{-in\tau}$$

calculate $x^\mu = \frac{1}{2\pi}\int_0^{2\pi} X^\mu(\sigma,0)\mathrm{d}\sigma, \quad p^\mu = \int_0^{2\pi}\Pi^\mu(\sigma,0)\mathrm{d}\sigma$ and

$$-i\frac{\sqrt{4\pi T}}{2\pi}\int_0^{2\pi} X^\mu(\sigma,0)e^{-in\sigma}\mathrm{d}\sigma = \frac{1}{n}\left(\alpha_n^\mu - \overline{\alpha}_{-n}^\mu\right)$$

$$\frac{1}{2\pi}\sqrt{\frac{4\pi}{T}}\int_0^{2\pi}\Pi^\mu(\sigma,0)e^{-in\sigma}\mathrm{d}\sigma = \alpha_n^\mu + \overline{\alpha}_n^\mu$$

3. Wave Equations and Solutions

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- With these formulas calculate (for example)

$$
\begin{aligned}
\{x^\mu, p^\nu\} &= \frac{1}{2\pi}\{\int_0^{2\pi} X^\mu(\sigma,0)\,\mathrm{d}\sigma, \int_0^{2\pi} \Pi^\nu(\sigma',0)\,\mathrm{d}\sigma'\} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma \int_0^{2\pi}\mathrm{d}\sigma'\{X^\mu(\sigma,0),\Pi^\nu(\sigma',0)\} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma \int_0^{2\pi}\mathrm{d}\sigma' g^{\mu\nu}\delta(\sigma-\sigma') = g^{\mu\nu}
\end{aligned}
$$

- In this way one finds ($\delta_{m+n} := \delta_{m+n,0}$)

$$
\begin{aligned}
\{\alpha_m^\mu, \alpha_n^\nu\} &= \{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\} = -im\delta_{m+n}g^{\mu\nu} \\
\{\overline{\alpha}_m^\mu, \alpha_n^\nu\} &= 0 \\
\{x^\mu, p^\nu\} &= g^{\mu\nu}
\end{aligned}
$$

3. Wave Equations and Solutions

3.6. The Witt Algebra

- Therefore using $L_m = \frac{1}{2}\sum_n \alpha_n \cdot \alpha_{m-n}$, $\overline{L}_m = \frac{1}{2}\sum_n \overline{\alpha}_n \cdot \overline{\alpha}_{m-n}$:

$$\boxed{\begin{gathered}\{L_m, L_n\} = -i(m-n)L_{m+n} \qquad \{\overline{L}_m, \overline{L}_n\} = -i(m-n)\overline{L}_{m+n} \\ \{L_m, \overline{L}_n\} = 0\end{gathered}}$$

- $\Rightarrow$**Witt algebra** ($\rightarrow$ quantize to get Virasoro algebra).
- The Virasoro modes $L_m, \overline{L}_m$ generate an infinite dimensional Lie algebra (Witt algebra) of conserved charges respecting the closed string boundary condition.

3. Wave Equations and Solutions

3.6. The Witt Algebra

- For the open string the Virasoro modes $L_m = \frac{1}{2}\sum_n \alpha_n \alpha_{m-n}$ fulfill the same commutation relation

$$\boxed{\{L_m, L_n\} = -i(m-n)L_{m+n}}$$

3. Wave Equations and Solutions

3.6. The Witt Algebra

- One question is left:
  Why did the infinite dimensional Witt algebra appear in this classical calculation?
- Consider the unit circle S^1 and the group of diffeomorphisms on it. A diffeomorphism $\theta \to \theta + a(\theta)$ is generated by the operator $D_a = ia(\theta)\frac{d}{d\theta}$.
  A complete basis for such operators is given by $D_n = ie^{in\theta}\frac{d}{d\theta}$ fulfilling the commutator relation

$$[D_n, D_m] = -i(m-n)D_{m+n}.$$

- We see: The Witt algebra is simply the **Lie algebra of the group of diffeomorphisms on the circle**!

Summary

We ...

- ... defined Nambu-Goto action in analogy to the point particle.
- ... found equivalent Polyakov action with 'nicer' properties.
- ... discussed symmetries and found equations of motion.
- ... used conformal gauge to simplify equations of motion.
- ... derived general solutions for open and closed strings.
- ... realized that the Virasoro modes generate the Witt algebra.

Questions?

Thank you for your attention!