The Conformal Group In Various Dimensions

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1 Introduction

This report is associated with the talk *The Conformal Group In Various Dimensions* held on 04.03.2013 in a proseminar series on Conformal Field Theory and String Theory at ETH Zurich. Ideas mentioned in the talk (which is available on www.itp.phys.ethz.ch) will be elaborated in the report and derivation are worked out more thoroughly with details and further thoughts given on the material presented.

Conformal field theories and in particular conformal quantum field theories are of high interest in modern theoretical physics. This is partly due to the possible applications among which there is a description of second order phase transitions and quantum behavior at very high energies. Both situations share scale invariance of space and mass respectively and thus are good examples of systems with symmetries exceeding the usual given Lorentz symmetry. The other important contribution of conformal theories to physics is as a playground for applying theoretical tools since those theories - exhibiting very high (and sometimes even infinite) amounts of symmetry are relatively simple to solve with basic mathematics; most notably group theory and complex analysis.

The ideas presented in this report will follow the same structure as the talk. Prerequisite material is quickly reviewed in the *Repetition and Noethers theorem* section (mostly from [2]) before deriving *The structure of infinitesimal conformal transformations* after which we are presented with the conformal group - essentially an enlarged Lorentz group. The notion of *Global conformal transformations* will be made clear when compared to the local conformal group - a difference only occurring in the important 2D-case (material from [3] is used). Things will be finished up by giving a short mathematical introduction to *The Virasoro algebra in 2 dimensions* , the unique central extension of the conformal algebra that arises naturally due to the quantum mechanical nature of the systems we study. This is also the first time that the discussion will be restricted to quantum mechanics. All other ideas presented could as well be true for classical field theories. Since the last section is rather mathematical, [1] is used as the primary source).

2 Repetition and Noether's Theorem

In this section we will revise the importance of symmetry in field theories. We know from Classical Mechanics that Lagrange's equations of motions

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \tag{1}$$

are equivalent to Hamilton's principle of least action

$$\delta S = 0 \tag{2}$$

where the action is by definition

$$S := \int d^d x L(x_i, \dot{x}_i, t) \tag{3}$$

In field theory we build upon this idea by switching from a discrete to a continuous set of parameters - the values of the fields and their first derivatives at any position. We now consider general transformations of the action functional (*functional* because it now depends on the fields Φ and $\partial_\mu \Phi$ which are now essentially functions) before going to symmetry transformations. In the talk we restricted ourselves to *continuous* transformations, i.e. transformations that are continuously connected to the identity by a small parameter ϵ. Discrete transformations like time are space inversion are neglected here because the all important Noethers theorem which will be stated shortly requires continuous transformations. A general transformation changes not only the coordinates but also the fields (figure 1):

$$x \to x' \tag{4}$$

$$\Phi(x) \to \Phi'(x') =: \mathcal{F}(\Phi(x)) \tag{5}$$

It is also to be stated that the same transformation can be understood in the so called active and passive sense. In the active view we actually perform a rotation for example on space-time. If you drew up a sample field on a sheet of paper this would amount to twisting the paper whereas in the passive view the observer goes around the sheet and thus only changes his point of view of the same thing. It is of course clear that we can regard every transformation as being generated by one or the other point of view but in we will stick with the active view for the time being.

2.1 Examples

Let us revise some basic examples.

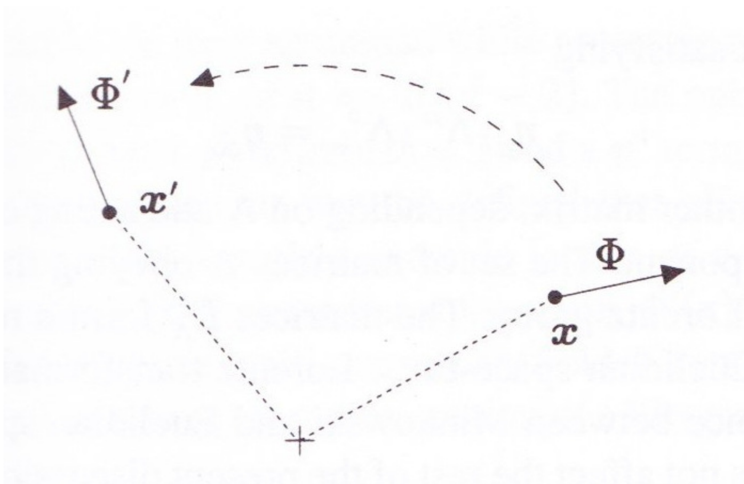

Figure 1: Active transformation [2]

2.1.1 Example 1 - Translations

Under a translation by a vector a the coordinates change according to

$$x' = x + a \tag{6}$$

In order to stay invariant under this coordinate only transformation the fields must change like

$$\Phi'(x + a) = \Phi'(x') = \Phi(x) \tag{7}$$

which leads us to field transformation function

$$\mathcal{F} = \text{Identity} \tag{8}$$

We are now interested in the variation of the action $\delta S = S - S'$. One can write down the general transformed action

$$S' = \int d^d\text{x} \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\Big(\mathcal{F}\big(\Phi(x)\big), \Big(\frac{\partial x^\nu}{\partial x'^\mu}\Big)\partial_\nu \mathcal{F}\big(\Phi(x)\big)\Big) \tag{9}$$

Since we additionally have

$$\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu \tag{10}$$

the Jacobian determinant is 1 and we obtain

$$S = S' \tag{11}$$

Note that the derivation inside the arguments of the Lagrangian is with respect to the spacial coordinates and not with respect to Φ, i.e. $\partial_\mu \mathcal{F}$ is not zero but $\partial_\mu \Phi(x)$

2.1.2 Example 2 - Lorentz Transformations

We now turn to Rotations of Minkowski space-time, i.e. Lorentz transformations. We are obliged to include Lorentz symmetry in our theories by special relativity. Under a Lorentz transformation by a Lorentz matrix Λ the coordinates change according to

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{12}$$

which gives

$$\frac{\partial x^\nu}{\partial x' \mu} = \left(\Lambda^{-1}\right)^\nu_\mu \tag{13}$$

The fields change linearly according to

$$\Phi'(\Lambda x) = L_\Lambda \Phi(x) \tag{14}$$

The transformation function is then just a matrix.

$$\mathcal{F} = L_\Lambda \tag{15}$$

If we now plug all this into (equation 9) we obtain

$$S' = \int d^d\mathrm{x}\, \mathcal{L}\Big(L_\Lambda \Phi, \Lambda^{-1}\partial\big(L_\Lambda \Phi(x)\big)\Big) \tag{16}$$

2.1.3 Example 3 - Scale Transformations

As a third and last example we now consider this new type of transformations. This is given by

$$x' = \lambda x \tag{17}$$

$$\frac{\partial x^\nu}{\partial x'^\mu} = \lambda^{-\Delta} \tag{18}$$

$$\Phi'(\lambda x) = \mathcal{F}(\Phi(x)) = \lambda^{-\Delta}\Phi(x) \tag{19}$$

$$S' = \lambda^d \int d^d\mathrm{x}\, \mathcal{L}\Big(\lambda^{-\Delta}\Phi, \lambda^{-1-\Delta}\partial_\mu \Phi\Big) \tag{20}$$

We have not yet done any statement about possible invariance of the fields under those transformations. There are certain requirements for $\mathcal{F}$ which potentially lead to invariance which we will study in the next chapters. But before we will introduce the notion of *infinitesimal transformations* .

2.2 Infinitesimal Transformations

As the transformations we consider are continuously connected to the identity we can linearize their effects, i.e. represent them by a Taylor series up to first oder about the identity. For a general transformation depending on a set of parameters ω_a this can be written down as

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\partial x^{\mu}}{\partial \omega_a} \tag{21}$$

where the Einstein sum convention is understood. The effect on the fields can be linearized in terms of the continuous, differentiable transformation function $\mathcal{F}$

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\partial \mathcal{F}}{\omega_a} \tag{22}$$

We also recall the definition of the *Generator* G_a of an infinitesimal transformation depending on parameters a which relates the changed and unchanged field at the *same* position.

$$\Phi'(x) - \Phi(x) =: -i\omega_a G_a \Phi(x) \tag{23}$$

$$= -\left(\frac{\partial \Phi}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial \omega_a} \omega_a - \frac{\partial \mathcal{F}}{\partial \omega_a} \omega_a \right) \tag{24}$$

The last expression originates from plugging in the first order Taylor expansion and the chain rule for $\Phi(x')$. We can easily see that for the examples covered in section (2.1,2.2,2.3) the generators are as follows: For a translation in the direction of μ we have

$$P_{\mu} = -i\partial\mu \tag{25}$$

since $\frac{\partial x^{\mu}}{\partial a_{\mu}} = 1$ and $\frac{\partial \mathcal{F}}{\partial a_{\mu}} = 0$. Things are little more complicated in the case of Lorentz transformations. If we write

$$x^{\mu} + \omega_{\rho\nu} g^{\rho\mu} x^{\nu} \tag{26}$$

with the metric tensor g we recall that $\omega_{\rho\nu} = -\omega_{\nu\rho}$ must be antisymmetric to keep the metric invariant. This allows us to write the transformation function as

$$\mathcal{F}(\Phi) = \left(1 - \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu}\right)\Phi \tag{27}$$

Finally plugging this into the formula for the generators we obtain

$$L^{\rho\nu} = i\big(x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}\big) + S^{\rho\nu} \tag{28}$$

In four-dimensional Minkowski space-time we have 16 generators. We also rediscover the classical angular momentum operator as the first term. By the same procedure it can be shown that the generator of scale transformations must be

$$D = -ix^\mu \partial_\mu \tag{29}$$

2.3 Conserved Currents

In classical mechanics every continuous symmetry was associated with a conserved quantity, i.e if the system looked the same under rotations it exhibited angular momentum conservation, invariance to translations resulted in momentum conservation. Time invariance was associated with the conservation of energy. Systems were called *integrable* or *solvable* if the number of conserved quantities matched the number of degrees of freedom. With the continuous set of parameters introduced our notion of a conserved quantity changes in favor of the *conserved current*

$$j_a^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\nu \Phi - \delta^\mu_\nu \mathcal{L}\right)\frac{\partial x^\nu}{\partial \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\frac{\partial \mathcal{F}}{\partial \omega_a} \tag{30}$$

This quantity is associated with an infinitesimal transformation depending on parameters a. The motivation of this definition is that the variation of the action can then exactly be cast into the form

$$\delta S = S - S' \tag{31}$$

$$= -\int d^d\mathrm{x}\; \partial_\mu j_a^\mu \omega_a \tag{32}$$

This would not be of any use if ω_a represented a discrete symmetry. This integral vanishing for one particular ω_a tells us nothing about the currents j_a. Only the fact that this integral vanishes *for every* ω_a results in

$$\partial_\mu j_a^\mu = 0 \tag{33}$$

by the variational principle. This is *Noether's Theorem* .

3 The structure of infinitesimal conformal transformations

The structure of the Poincar group i.e. the Lorentz transformations plus translations is well known and has been revisited in the preceding section.

We want to elaborate on how allowing a scale factor enlarges the Poincar group and by doing so will discover a surprise, namely that the addition of scale transformations allows for a fourth kind of transformations to be added to the group structure.

3.1 Requirements for conformal invariance

By definition, conformal transformations only preserve angles (figure 2) (unlike Lorentz transformations which also keep lengths). This is equivalent to

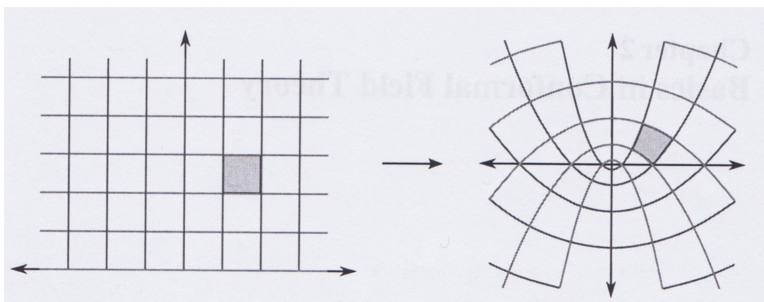

Figure 2: Conformal transformation[2]

leaving the metric tensor invariant up to a scaling factor $\Lambda(x)$ which can depend on position.

$$g_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu} \tag{34}$$

We will again follow the approach of infinitesimal transformations which allows us to compose finite transformations from the generators. Consider hence an infinitesimal map of the form

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho} + O(\epsilon^2) \tag{35}$$

and plug this into the constraint equation (equation 34) we can place a constraint on the ϵ^{ρ}s after a short calculation.

$$\frac{\partial x'^{\rho}}{\partial x^{\mu}} = \delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho} \tag{36}$$

$$\Lambda(x) g_{\mu\nu} = g_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} + O(\epsilon^2) \tag{37}$$

$$= g_{\rho\sigma}\left(\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho}\right)\left(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma}\right) + O(\epsilon^2) \tag{38}$$

$$= g_{\rho\sigma}\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\rho}_{\mu}\partial_{\nu}\epsilon^{\sigma} + \partial_{\mu}\epsilon^{\rho}\delta^{\sigma}_{\nu}\right) + O(\epsilon^2) \tag{39}$$

$$= g_{\mu\nu} + \partial_{\nu}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\nu} \tag{40}$$

where we have lowered the indices of ϵ upon contraction with the metric tensor. To keep invariance up to scaling we naturally require

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = K g_{\mu\nu} \tag{41}$$

where K can be determined by tracing with $g_{\mu\nu}$

$$g^{\mu\nu}\big(\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu\big) = K g_{\mu\nu} g^{\mu\nu} \tag{42}$$

$$2(\partial \cdot \epsilon) = dK \tag{43}$$

where we introduced $\partial \cdot \epsilon := \partial_\nu \epsilon^\nu$ and d is the dimension of the metric tensor. Going back to equation (40) we conclude that

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu} \tag{44}$$

which we refer to as the *constraint equation* for conformal transformations. Every transformation of the form (35) has to fulfill (44). We can also derive

$$(d-1)\partial^\mu \partial_\mu (\partial \cdot \epsilon) = 0 \tag{45}$$

if d is *not* equal to 2.

3.2 The four types

If we look at equation (45) we find that ϵ_μ is constrained to be at most quadratic in the position arguments. Any cubic or higher dependence would violate the third derivative being zero everywhere (bare in mind however that (45) was derived for the case $d \neq 2$!). By this token we can make the ansatz

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{46}$$

Now the constraint equation (34) is true everywhere in space. This means that we can consider the constant, linear and quadratic term separately, place constraints on the coefficients, extract the according generators and recompose any arbitrary transformation in the end.

3.2.1 The constant term - Translations

By this logic we consider a constant $\epsilon_\mu = a_\mu$ first. From equation (44) we get

$$\partial_\nu a_\mu + \partial_\mu a_\nu = \frac{2}{d}(\partial \cdot a) g_{\mu\nu} \tag{47}$$

and since a is constant we trivially get $0 = 0$. The variable a is hence totally arbitrary. The transformations associated with $x' = x + a$ are of course the translations and their generators were determined to be

$$P_\mu = -i\partial_\mu \tag{48}$$

in section (2.1).

3.2.2 The linear term - Dilations and Rotations

Consider now the linear term only

$$\epsilon_\mu = b_{\mu\nu}x^\nu \tag{49}$$

By equation (44) we get

$$\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu = \frac{2}{d}(\partial\cdot\epsilon)g_{\mu\nu} \tag{50}$$

$$\partial_\mu b_{\nu\rho}x^\rho + \partial_\nu b_{\mu\rho}x^\rho = \frac{2}{d}\partial_\sigma b_{\sigma\rho}x^\rho g_{\mu\nu} \tag{51}$$

$$b_{\nu\rho}\delta^\rho_\mu + b_{\mu\rho}\delta^\rho_\nu = \frac{2}{d}b_{\sigma\rho}g_{\mu\nu}\delta^\rho_\sigma \tag{52}$$

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}b^\rho_\rho g_{\mu\nu} \tag{53}$$

which tells us that we can decompose the second-rank tensor $b_{\nu\mu}$ into an antisymmetric part (for $\mu \neq \nu$, $g_{\mu\nu}$ will be zero since the metric tensor we consider is diagonal) and a multiple of the identity matrix ($\mu = \nu$). We recover dilations in the latter case since the transformation then reads

$$x'^\mu = \alpha x^\mu \tag{54}$$

and the associated generator was given in equation (29). An antisymmetric matrix acting as infinitesimal parameters of a transformation is familiar since section (2.1.2). We identify those with the usual Lorentz transformations.

3.2.3 The quadratic term - Special Conformal Transformations

It was to be expected that dilations arise naturally if $\Lambda \neq 1$ is allowed in addition to the Poincar transformations. However we are not done yet! Pluggin in $\epsilon_\mu = c_{\mu\nu\rho}$ into equation (44) we obtain

$$c_{\mu\nu\rho} = g_{\mu\rho}c^\sigma_{\sigma\nu} + g_{\mu\nu}c^\sigma_{\sigma\rho} - g_{\nu\rho}c^\sigma_{\sigma\mu} \tag{55}$$

It becomes clear in this form that there are only d degrees of freedom for this transformation and not $3d$ as could have been expected. Casting those d parameters into a vector b it is possible to show that the according transformation takes the form

$$x'^{\mu} = \frac{x^{\mu} - (x \cdot x)b^{\mu}}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} \tag{56}$$

The generators of this are found to be

$$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - (x \cdot x)\partial_{\mu}) \tag{57}$$

The meaning of such a transformation can be understood by considering

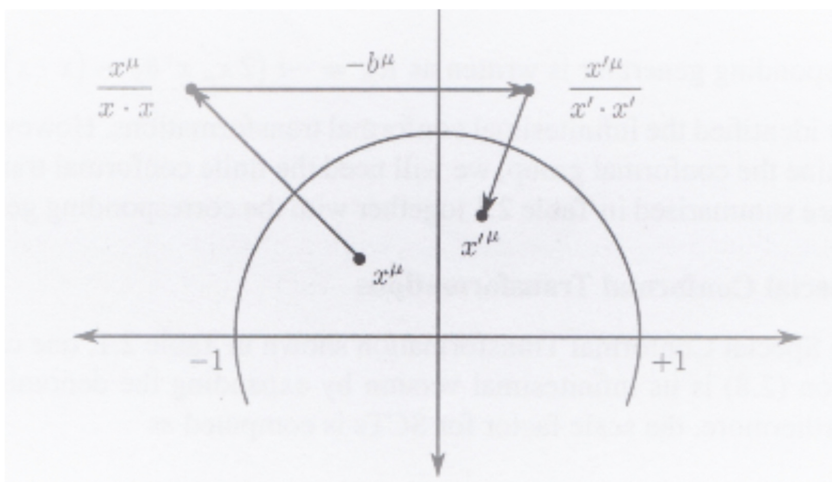

Figure 3: Special conformal transformation [2]

figure (3). It is actually an inversion on the unit circle then a translation by the vector b and then another inversion. The astout reader may ask why this transformation did not arise before allowing a scale factor. The answer is that only in these conditions the transformation containing an inversion can be continuously connected to the identity. For our further discussion we should also be aware of the fact that the point

$$x'^{\mu} = \frac{1}{b \cdot b} b^{\mu} \tag{58}$$

is mapped to infinity.

4 Global Conformal Transformations

In this section we will consider the globally defined versions of the transformations found in section (2). In the end an interesting identification can be made namely that the group of all globally defined conformal transformations is isomorphic to the Lorentz group in d+2 dimensions. The case $d = 2$ - which was excluded in order to use equation (45) and motivate our ansatz - will also be reviewed in detail.

4.1 The group SO(p,q)

The group SO(p) is known from linear algebra; it is just the group of orthogonal matrices with determinant one, i.e. they leave the scalar product

$$< x, y >= x^T g y \tag{59}$$

invariant, where g in this case is just the identity matrix. The generalization to other metric tensors is straightforward: Members of the group SO(p,q) leave invariant the scalar product

$$< x, y >= x^T g y \tag{60}$$

where g is the metric tensor with p +1s and q −1s on the diagonal. The most familiar example is the Lorentz group SO(3,1). In depth calculations show that the commutation relations of the SO(p,q) group are

$$[J_{ab}, J_{cd}] = i(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} + g_{bd}J_{ac}) \tag{61}$$

The generators of the conformal transformations represent the conformal group and they were calculated in equations (25), (28), (29) and (57). Their commutation relations read

$$[D, P_\mu] = iP_\mu \tag{62}$$
$$[D, K_\mu] = -iK_\mu \tag{63}$$
$$[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - L_{\mu\nu}) \tag{64}$$
$$[K_\rho, L_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu \tag{65}$$
$$[P_\rho, L_{\mu\nu}] = i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu \tag{66}$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} + g_{\mu\rho}L_{\nu\sigma} + g_{\nu\sigma}L_{\mu\rho}) \tag{67}$$

The generators can be cast into a matrix $J_{\mu\nu}$ in a particular way. We define

$$J_{\mu\nu} := L_{\mu\nu} \tag{68}$$

$$J_{-1,\mu} := \frac{1}{2}(P_\mu - K_\mu) \tag{69}$$

$$J_{-1,0} := D \tag{70}$$

$$J_{0,\mu} := \frac{1}{2}(P_\mu + K_\mu) \tag{71}$$

Again the commutation relations can be computed and the result is

$$[J_{ab}, J_{cd}] = i(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} + g_{bd}J_{ac}) \tag{72}$$

Compare equations (61) and (72). They are equal. Identifying the matrix elements we have contracted a one to one correspondence between the elements of SO(d+1,1) and the conformal group in d dimensions. Quickly consider the dimensions. The group SO(d+1,1) is completely determined by its' upper triangular part (consider the Lorentz group to see this). In $d+2$ dimensions this amounts to $\sum_{n=1}^{d+2} n = \frac{(d+2)(d+1)}{2}$ elements. Now count the number of independent generators of the conformal group. There are d generators for both translations and special conformal transformations, rotations (with their constraint of antisymmetry) add another $\frac{d(d-1)}{2}$ while dilations only have one generator. We quickly calculate

$$d + d + \frac{d(d-1)}{2} + 1 = \frac{(d+2)(d+1)}{2} \tag{73}$$

Hence the conformal group in d dimensions is isomorphic to the group SO(d+1,1) with $\frac{(d+2)(d+1)}{2}$ parameters.

4.2 The d=2 case

Applying the aforementioned to the two-dimensional case we are interestingly have

$$\text{conformal group} \approx SO(3,1) \approx SL(2,\mathbb{C}) \tag{74}$$

This is indeed true for the *global* conformal group for $d = 2$ (we have not assumed equation (45) in section (4.1)) but we expect the infinitesimal structure to be quite different since there the failure of (45) to hold will have large consequences. What is still true is equation (44). For $\mu = \nu = 0, 1$ we get in both cases

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1 \tag{75}$$

For $\mu \neq \nu$ we are on the off-diagonal of the metric tensor (which we just take to be the identity for a moment - no Minkowski space involved) and get a zero:

$$\partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \tag{76}$$

If we identify $\epsilon_{0,1}$ with the real and imaginary part of a complex function respectively we recover the *Cauchy-Riemann equations*! This means that the infinitesimal conformal maps are exactly the holomorphic ones on the complex plane. A general locally holomorphic map can be written down as

$$f(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}) \tag{77}$$

which can be decomposed into functions $f_n = \epsilon_n(-z^{n+1})$ corresponding to the summands in the expansion. Each of those infinitesimal transformations has a corresponding generator according to (24). Computing $\frac{\partial x^\mu}{\partial \omega^a} = \frac{\partial z'}{\partial \epsilon_n} = -z^{n+1}$ we obtain

$$l_n := -z^{n+1} \partial_z \tag{78}$$

We compute the commutation relation of those generators

$$[l_m, l_n] = -z^{m+1} \partial_z (z^{n+1} \partial_z) + z^{n+1} \partial_z (z^{m+1} \partial_z) \tag{79}$$

$$= (m-n) l_{m+n} \tag{80}$$

$$\tag{81}$$

Those generators constituting any Laurent series together with this commutation relation are called *Witt Algebra* . Obviously, it is infinite-dimensional. But wait! Shouldn't the group of conformal transformations in d dimensions be isomorphic to $SO(d+1, 1)$? For the case $d = 2$ this would only allow for six dimensions, not infinitely many. This paradox comes from the fact that (45) was used to derive the correlation between those groups but it does not hold for the infinitesimal transformations considered here. Let us instead review global conformal transformations, i.e. those which map the Riemann sphere in a 1-1 and holomorphic way onto itself. From complex analysis we know that those functions have to fulfill a couple of constraints.

- It may not have essential singularities
- Injectivity requires that it may only have one pole of order one and one zero of multiplicity one. Hence only functions of the form

  $$f(z) = \frac{p(z)}{q(z)} \tag{82}$$

  are possible.

- It must satisfy the group property, i.e. the combination must again be such a transformation.

The only holomorphic function left are thus

$$f(z) = \frac{az+b}{cz+d} \qquad \text{with } ad - bc = 1 \tag{83}$$

where 1 is a constant chosen for simplicity. Any other constant would be equivalent up to rescaling. Those transformations are called *Mobius Transformations* We now consider the combinations of such a Mobius Transformation F with parameters a, b, c and d with a second one G with the parameters e, f, g and h:

$$\begin{aligned} F(G(z)) &= \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} && (84) \\ &= \frac{aez + af + bgz + bh}{cez + cf + dgz + dh} && (85) \\ &= \frac{z(ae+bg) + (af+bh)}{z(ce+dg) + (cf+dh)} && (86) \\ &=: \frac{Az+B}{Cz+D} && (87) \end{aligned}$$

where we have defined

$$\begin{aligned} A &:= ae + bg && (88) \\ B &:= af + bh && (89) \\ C &:= ce + dg && (90) \\ D &:= cf + dh && (91) \end{aligned}$$

We see that indeed the composition of two Mobius Transformations results in another Mobius Transformation. What's more, the structure of the resulting coefficients lets us identify the Mobius composition with a matrix multiplication

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{92}$$

Together with the constraint on what we now see turns out to be the determinant of the according matrix we can identify every Mobius transform with an element of $SL(2,\mathbb{C})$ which is in turn isomorphic to $SO(3,1)$ which we know from linear algebra. We have thus shown that for global conformal transformations, we recover the behavior of the transformation group elaborated in section 4.1.

That still does not answer what is wrong with the Witt Algebra in the first place. Let us reconsider the generators l_n given by (**??**generatorl). We first note that the principal part of the Laurent series diverges at $z = 0$ for $n < -1$. Now we study the behavior of the side part for $z \to \infty$. Again from complex analysis we know that $f(z)$ is defined to have a singularity at $z = \infty$ if $f(w := \frac{1}{z})$ has a singularity at $w = 0$. Thus we first have to recast the generators into that shape. Since

$$z = \frac{1}{w} \tag{93}$$

$$\to \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \tag{94}$$

$$= -\frac{1}{z^2}\frac{\partial}{\partial w} \tag{95}$$

$$= -w^2\frac{\partial}{\partial w} \tag{96}$$

we have that

$$l_n = -z^{n+1}\partial_z \tag{97}$$

$$= -\Big(-\frac{1}{w}\Big)^{n-1}\partial_w \tag{98}$$

A singularity at $w = 0$ in the latter expression only occurs for $n > 1$. The only generators that are thus allowed in a holomorphic 1-1 Laurent series are l_{-1}, l_0 and l_1. It remains to be said that the local and global perspective being inequivalent is a particularity of the two-dimensional case.

5 The Virasoro Algebra $\mathfrak{Vir}$ in two dimensions

In section 4.2 we found out that the *Witt Algebra* finds realization in classical conformal field theories. Due to quantum effects, we have to consider any *central extension* that the algebra might exhibit. A central extension of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ is an exact sequence of Lie algebra homomorphisms

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0 \tag{99}$$

where $[\mathfrak{a}, \mathfrak{h}] = 0$ and the image of every homorphism is the kernel of the succeeding one. One can show that for every such sequence there is a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$. Now let $\Theta : \mathfrak{g} x \mathfrak{g} \to \mathfrak{a}$

$$\Theta(X, Y) := [\beta(X), \beta(Y)] - \beta([X, Y]) \tag{100}$$

This bilinear form is obviously alternating and it fulfills a quasi-Jacobi identity, namely

$$\Theta(X, [Y, Z]) + \Theta(Y, [Z, X]) + \Theta(Z, [X, Y]) = 0 \tag{101}$$

This - a bilinear, alternating map fulfilling the last identity - is what we call a *cocycle* . It is an algebraic theorem that every central extension has exactly one associated cocycle and it is *trivial* if there is a homomorphism μ such that

$$\Theta(X, Y) = \mu([X, Y]) \tag{102}$$

for all X and Y. After those definitions we turn back to our case of the *Witt Algebra* . We stare at equation (99) and identify

$$\mathfrak{a} \leftrightarrow \mathbb{C} \tag{103}$$

$$\mathfrak{h} \leftrightarrow \mathfrak{Vir} \tag{104}$$

$$\mathfrak{g} \leftrightarrow \mathfrak{W} \tag{105}$$

And educated guess would then tell us that the cocycle defining the only nontrivial central extension of $\mathfrak{W}$ by $\mathbb{C}$ is

$$\omega(L_n, L_m) := \delta_{n+m,0} \frac{n}{12}(n^2 - 1) \tag{106}$$

It remains to be shown that this is indeed a cocycle, that it is non-trivial and that it is indeed the only cocycle that can be associated with the *Witt Algebra* .

The proposed cocycle is evidentially alternating and bilinear. It also fulfills

$$\omega(L_k, [L_m, L_n]) + \omega(L_m, [L_n, L_k]) + \omega(L_n, [L_k, L_m]) = 0 \tag{107}$$

hence it indeed is a cocycle. Is it trivial? Consider a homomorphism μ that fulfills equation (102). Plugging in L_0 we get

$$\mu(L_0) = \frac{1}{24}(n^2 - 1) \tag{108}$$

in which the left side is independent of n whereas the right side is not. The cocylce ω is thus non-trivial. The proof of uniqueness involves more detailed calculations and can for example be found in [1]. By the above, ω defines the unique central extension of the *Witt Algebra* in the following way. Set

$$\mathfrak{Vir} = \mathfrak{W} \oplus \mathbb{C} \tag{109}$$

and complete the definition by specifying the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + c\frac{n}{12}(n^2 - 1)\delta_{m+n,0} \tag{110}$$

6 References

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