CFT - Basic properties and examples

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- Structure of correlation functions
- Energy momentum tensor in conformal field theory
- The free Boson and Fermion in 2 dimensions with calculation of the central charge

2D CFT basic definitions

chiral and antichiral fields

Fields depending only on z are chiral fields, fields depending only on $\overline{z}$ are called antichiral fields.

conformal dimensions

property of field under scalings $z \mapsto \lambda z$

$$\phi(z, \overline{z}) \mapsto \lambda^h \overline{\lambda}^{\overline{h}} \phi(\lambda z, \overline{\lambda} \overline{z})$$

primary fields

conformal transformation $z \mapsto f(z)$

$$\phi(z, \overline{z}) \mapsto \phi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi(f(z), \overline{f}(\overline{z}))$$

Structure of correlation functions

Definition

$$\langle \mathcal{T}\left(\phi(t_1)\phi(t_2)\ldots\phi(t_N)\right)\rangle = \frac{\int[d\phi]\phi(t_1)\phi(t_2)\ldots\phi(t_N)\exp\left(iS_\varepsilon[\phi(t)]\right)}{\int[d\phi]\exp\left(iS_\varepsilon[\phi(t)]\right)}$$

However this only makes sense if the Operators are orderd in time!

Two point function under conformal invariance

$$\langle \phi_1(z)\phi_2(\omega) \rangle = g(z,\omega)$$

with ϕ_1, ϕ_2 quasi-primary fields:

- Translation invariance
- Invariance under rescalings
- Invariance under inversion

thus:

$$\langle \phi_i(z)\phi_j(\omega) \rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z-\omega)^{2h_i}}$$

Three point function under conformal invariance

Same steps of derivation lead to this expression:

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}}$$

Infinitesimal conformal transformation

Consider conformal trafo $f(z) = z + \varepsilon(z)$ with $\varepsilon << 1$. Change of a primary field:

$$\delta_{\varepsilon,\overline{\varepsilon}}\phi(z,\overline{z}) = \left(h\partial\varepsilon(z) + \varepsilon(z)\partial + \overline{h\partial}\overline{\varepsilon}(\overline{z}) + \overline{\varepsilon}(\overline{z})\overline{\partial}\right)\phi(z,\overline{z})$$

Energy momentum tensor in conformal field theories

Noether

$\delta S = 0$ conserved current j from infinitessimal transformation
$x'^{\mu} = x + \epsilon\omega^{\mu}$

$$0 = L\left(\phi(x'), \frac{\partial\phi}{\partial x'^{\nu}}, x'\right) d^d x' - L\left(\phi, \frac{\partial\phi}{\partial\phi}, x\right) d^d x$$

$$0 = \epsilon\partial_{\mu}\left(\eta^{\mu\nu} L\omega_{\nu} - \omega\partial\phi\frac{\partial L}{\partial\partial_{\mu}\phi}\right) d^d x$$

Energy momentum tensor in conformal field theories

Conserved current

$$j^{\mu} = \eta^{\mu\nu} L \omega_{\nu} - \omega_{\nu} \partial^{\nu} \phi \frac{L}{\partial(\partial_{\mu}\phi)}$$

Definition of the Energy momentum tensor $T^{\mu\nu}$

$$j^{\mu} = T^{\mu\nu} \omega_{\nu}$$

Energy momentum tensor in conformal field theories

What implications does conformal invariance have on the energy momentum tensor?

- $\partial_\mu T^{\mu\nu} = 0$
- $T^{\rho\nu} = T^{\nu\rho}$
- $T^\mu_\mu = 0$

Energy momentum tensor in 2D

Transformation to complex coordinates:

$$T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right)$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right)$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4} T^{\mu}_{\mu} = 0$$

Energy momentum tensor

In two dimensions one will get a chiral and an antichiral field:
$2\pi T_{zz}(z,\bar{z}) = T(z)$, $2\pi \overline{T}_{\bar{z}\bar{z}}(z,\bar{z}) = \overline{T}(\bar{z})$

Radial ordering 2D

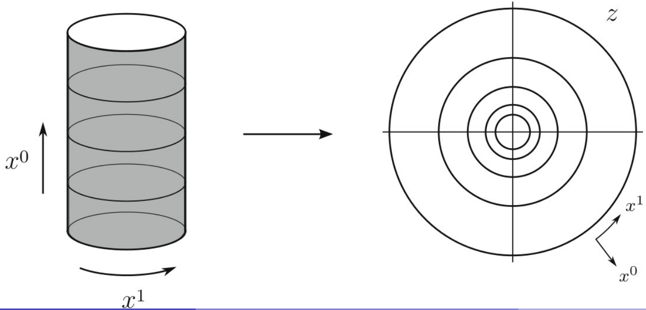
Compactification

Two variables x^0 for time and x^1 for space. Mapping of space variable onto circle.

Introduction of complex variables $\omega = x^0 + ix^1$

Mapping from the Cylinder to the Complex Plane

Mapping funciton: $z = e^{\omega} = e^{x^0} \cdot e^{ix^1}$



Conserved Charges

Definition

$$Q = \int dx^1 j_0 = \frac{1}{2\pi i} \oint_C \left(dz T(z)\epsilon(z) + d\overline{z}\,\overline{T}(\overline{z})\overline{\epsilon}(\overline{z}) \right)$$

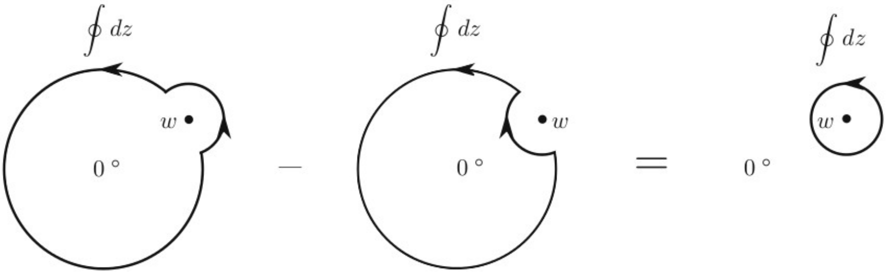
at $x^0 = \text{const}$ and $j_\mu = T_{\mu\nu}\epsilon^\nu$

From QFT we know that:

$$\delta A = [Q, A]$$

Radial ordering 2D

$$\delta_{\varepsilon,\overline{\varepsilon}}\phi(\omega,\overline{\omega}) = \frac{1}{2\pi i}\oint_{|z|>|\omega|} dz\varepsilon(z)T(z)\phi(\omega,\overline{\omega}) - \oint_{|z|<|\omega|} dz\varepsilon(z)\phi(\omega,\overline{\omega})T(z)$$
$$= \oint_{C(\omega)} dz\varepsilon(z)\mathcal{R}\left([T(z),\phi]\right)$$



Operator Product expansion

Comparison to direct calculation of primary field yields this:

$$\mathcal{R}\left(T(z)\phi(\omega,\overline{\omega})\right) = \frac{h}{(z-\omega)^2}\phi(\omega,\overline{\omega}) + \frac{\partial_\omega}{z-\omega}\phi(\omega,\overline{\omega})$$

Definition

A field is called primary if the operator product expansion between $T(z)$ and $\phi(z,\overline{z})$ is of the above form.

The free boson

Action of a free boson

$$S = \kappa \int dz d\bar{z} \partial X(z,\bar{z}) \bar{\partial} X(z,\bar{z})$$

Variation

$$\partial \bar{\partial} X(z,\bar{z}) = 0$$

two point function

$$\langle x(z,\bar{z}) x(\omega,\bar{\omega}) \rangle = -\frac{1}{4\pi\kappa} \ln(z-\omega)$$

Energy Momentum Tensor

normal ordering of energy momentum tensor

$$T(z) = 2\pi\kappa : \partial X \partial X := 2\pi\kappa \lim_{z \to w} \left(\partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle \right)$$

conformal dimension of ∂x

$$T(z)\partial x(\omega) = \frac{\partial x}{(z-\omega)^2} + \frac{1}{z-\omega}\partial^2 x$$

Thus the conformal dimension is $h = 1$

Wick's Theorem

From Quantum field theory:

Contraction

$$: \phi_1 \phi_2 \phi_3 \phi_4 :=: \phi_1 \phi_3 : \langle \phi_2 \phi_4 \rangle$$

Wick's Theorem

A time ordered product is equal to the normal ordered product, plus all possible contractions.

Asymptotic states

Consider Laurant expansion of the primary function $\phi(z, \overline{z})$:

$$\phi(z, \overline{z}) = \sum_{n,\overline{m} \in \mathbb{Z}} z^{-n-h} \overline{z}^{-\overline{m}-\overline{h}} \phi_{n,\overline{m}}$$

Definition

Take a look at the infinite past: $|\phi\rangle = \lim_{z,\overline{z} \to 0} \phi(z, \overline{z}) |0\rangle$

Singularity

We want the equation to be non singular i.e. well defined at $z = 0$ thus $\phi_{n,\overline{m}} |0\rangle = 0$ for $n > -h$ or $\overline{m} > -\overline{h}$

Out state

The same can be done with the out state. For $n < h$ or $\overline{m} < \overline{h}$

$$\langle 0| \phi_{n,\overline{m}} = 0$$

Central charge of the boson

Virasoro Algebra

Extension of the Witt Algebra with following commutation relation:

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$$\frac{c}{2} = \langle L_2 L_{-2} \rangle$$

Central charge of free boson

This calculation gives us a central charge of $c = 1$

The free fermion

Action of a free fermion

$$S = \kappa \int dz d\overline{z} (\psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi})$$

Variation

$$\partial \overline{\psi} = \overline{\partial} \psi = 0$$

Two point function

$$S = \frac{1}{2}\int dx^2 dy^2 \psi_i(x) A_{ij}(x,y)\psi_j(y)$$

With $A_{ij} = \kappa 2\pi\delta(x-y)(\gamma^0\gamma^\mu)_{ij}\partial_\mu$

$$\langle \psi_i(x)\psi_j(y)\rangle = A_{ij}^{-1}$$

two point function

$$\langle \psi(z)\psi(\omega)\rangle = -\frac{1}{2\pi\kappa}\frac{1}{z-\omega}$$

Energy momentum tensor

Energy momentum tensor

$$T(z) = -\pi\kappa : \psi(z)\partial\psi(z) :$$

conformal charge of ψ

$$T(z)\psi(\omega) = \frac{1}{2(z-\omega)^2}\psi(\omega) + \frac{1}{z-\omega}\partial_\omega\psi(\omega)$$

Thus $\psi(w)$ is a field of conformal dimension: $h = \frac{1}{2}$

Central charge of a free fermion

Same calculation as for the boson gives the central charge for the free fermion.

$$
\begin{aligned}
\frac{c}{2} = \langle 0|L_2 L_{-2}|0\rangle = \frac{1}{(2\pi i)^2}\oint dz \oint d\omega \frac{z^3}{\omega}\langle 0|T(z)T(\omega)|0\rangle = \\
(\pi\kappa)^2 \oint\oint \frac{dz d\omega z^3}{(2\pi i)^2\omega}\cdot \\
\cdot\left(\langle\psi(z)\partial\psi(\omega)\rangle\langle\partial\psi(z)\psi(\omega)\rangle + \langle\psi(z)\psi(\omega)\rangle\langle\partial\psi(z)\psi(\omega)\rangle\right) = \\
\frac{1}{4}\frac{1}{(2\pi i)^2}\oint dz \oint d\omega \frac{z^3}{\omega}\frac{1}{(z-\omega)^4} = \frac{1}{4}
\end{aligned}
$$

Central charge of free fermion

The central charge is $c = \frac{1}{2}$