Boundary Conformal Field Theory

Zhuli HE
ETH Zurich

Topics

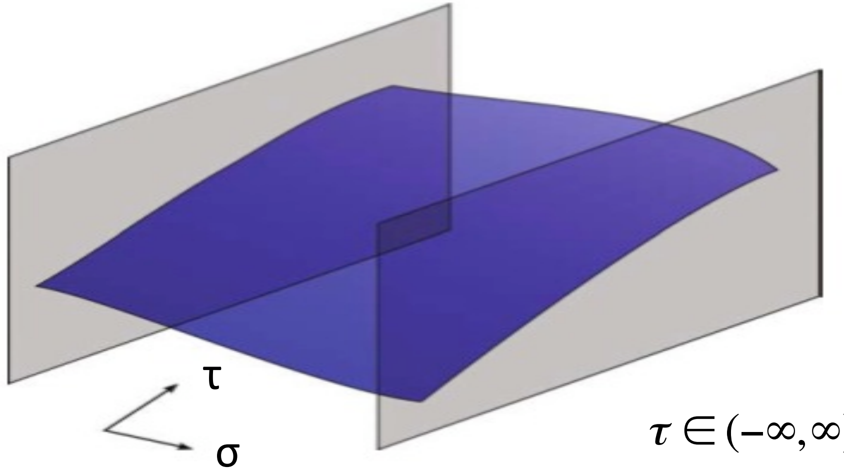
- Part I: Boundary Conditions & Boundary States
  - 1.1 Conformal Invariance & Boundary Conditions
  - 1.2 Boundary States & Gluing Condition
  - 1.3 Partition function & Loop-channel – Tree channel Equivalence
- Part II: Cardy condition
- Part III: g-function

1.1.1 Least Action Principle in Presence of Boundaries

Consider the 2D action for a free boson *X(τ ,σ)*

$$S = \frac{1}{4\pi}\int d\sigma\, d\tau((\partial_\sigma X)^2 + (\partial_\tau X)^2)$$

where the boundary term will be taken into account.



$\tau \in (-\infty, \infty)$

$\sigma \in [0, \pi]$

2D surface with boundaries.

1.1.1 Least Action Principle in Presence of Boundaries

$$\delta_X S = 0 \quad for \quad S = \frac{1}{4\pi}\int d\sigma\, d\tau((\partial_\sigma X)^2 + (\partial_\tau X)^2)$$

$$\delta_X S = \frac{1}{\pi}\int d\sigma\, d\tau\big((\partial_\sigma X)(\partial_\sigma \delta X) + (\partial_\tau X)(\partial_\tau \delta X)\big)$$

$$= \frac{1}{\pi}\int d\sigma\, d\tau\Big(-\big(\partial_\sigma^2 + \partial_\tau^2\big)X\cdot\delta X + \partial_\tau\big(\partial_\tau X\cdot\delta X\big) + \partial_\sigma\big(\partial_\sigma X\cdot\delta X\big)\Big)$$

The first term leads to KG-Eqn., and the two remaining term can be written as:

$$\delta_X S = \frac{1}{\pi}\int d\sigma\, d\tau\Big(\partial_\tau\big(\partial_\tau X\cdot\delta X\big) + \partial_\sigma\big(\partial_\sigma X\cdot\delta X\big)\Big)$$

$$= \frac{1}{\pi}\int d\sigma\, d\tau\vec{\nabla}\cdot\big(\vec{\nabla}X\delta X\big)$$

$$= \frac{1}{\pi}\int_B dl_B\big(\vec{\nabla}X\cdot\vec{n}\big)\delta X = 0$$

$\Rightarrow$

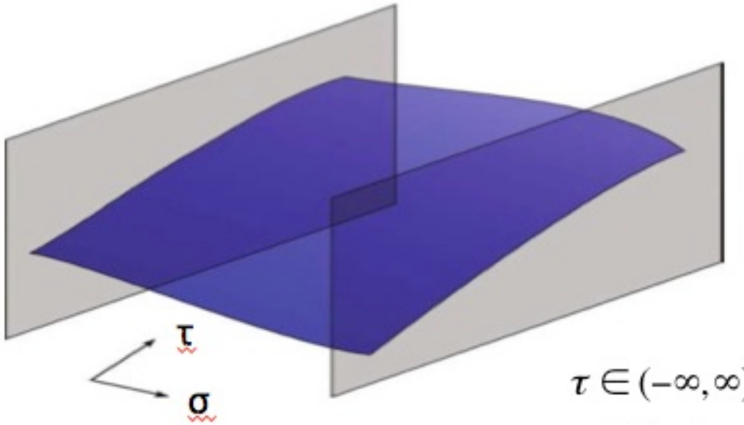
$$0 = \frac{1}{\pi}\int d\tau(\partial_\sigma X)\delta X\Big|_{\sigma=0}^{\sigma=\pi}$$

1.1.1 Least Action Principle in Presence of Boundaries

$$0 = \frac{1}{\pi}\int d\tau (\partial_\sigma X)\delta X\Big|_{\sigma=0}^{\sigma=\pi}$$

This equation allows two different solutions,
Hence two different boundary conditions:

$\partial_\sigma X\big\|_{\sigma=0,\pi} = 0$	Neumann condition
$\delta X\big\|_{\sigma=0,\pi} = 0 = \partial_\tau X\big\|_{\sigma=0,\pi} = 0$	Dirichlet condition

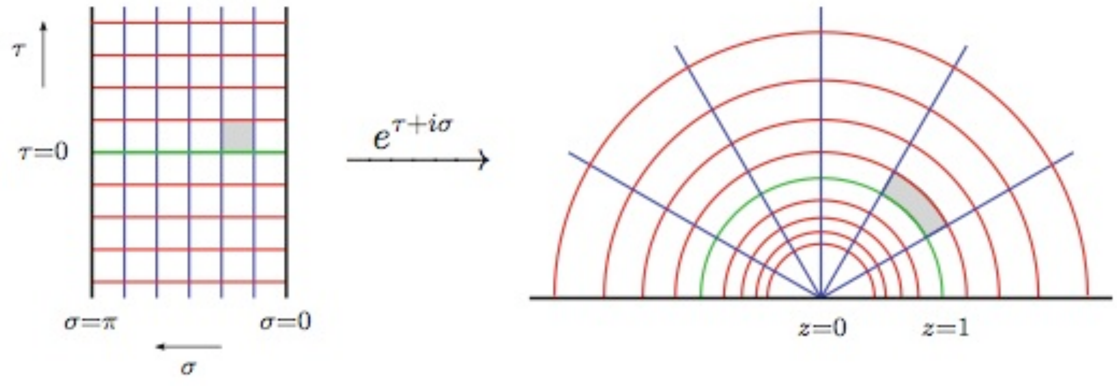


$\tau \in (-\infty, \infty)$

$\sigma \in [0, \pi]$

1.1.2 Boundary conditions for the Laurent modes

Consider this mapping:



Then express the boundary conditions in terms of the Laurent modes.

Recall $j(z) = i\partial X(z,\bar{z})$ we find

$$\partial_\sigma X\Big| = i(\partial - \bar{\partial})X = j(z) - \bar{j}(\bar{z}) = \sum_{n\in Z}(j_n z^{-n-1} - \bar{j}_n \bar{z}^{-n-1})$$

$$i\cdot\partial_\tau X\Big| = i(\partial + \bar{\partial})X = j(z) + \bar{j}(\bar{z}) = \sum_{n\in Z}(j_n z^{-n-1} + \bar{j}_n \bar{z}^{-n-1})$$

Apply to the boundary condition:

$j_n - \bar{j}_n = 0$		Neumann condition
$j_n + \bar{j}_n = 0$	$(\pi_0 = 0)$	Dirichlet condition

1.1.3 Conformal Symmetry

- Laurent modes of two currents $j(z), \quad \bar{j}(\bar{z})$ we have

$$j_n - \bar{j}_n = 0 \quad N.C$$
$$j_n + \bar{j}_n = 0 \quad D.C$$

- Now, also consider the Conformal Symmetry generated by EM-tensor,

$$T(z) = \frac{1}{2} N(jj)(z), \quad \bar{T}(\bar{z}) = \frac{1}{2} N(\overline{jj})(\bar{z})$$

where the Laurent modes for EM-tensor $L_n = \frac{1}{2} N(jj)$

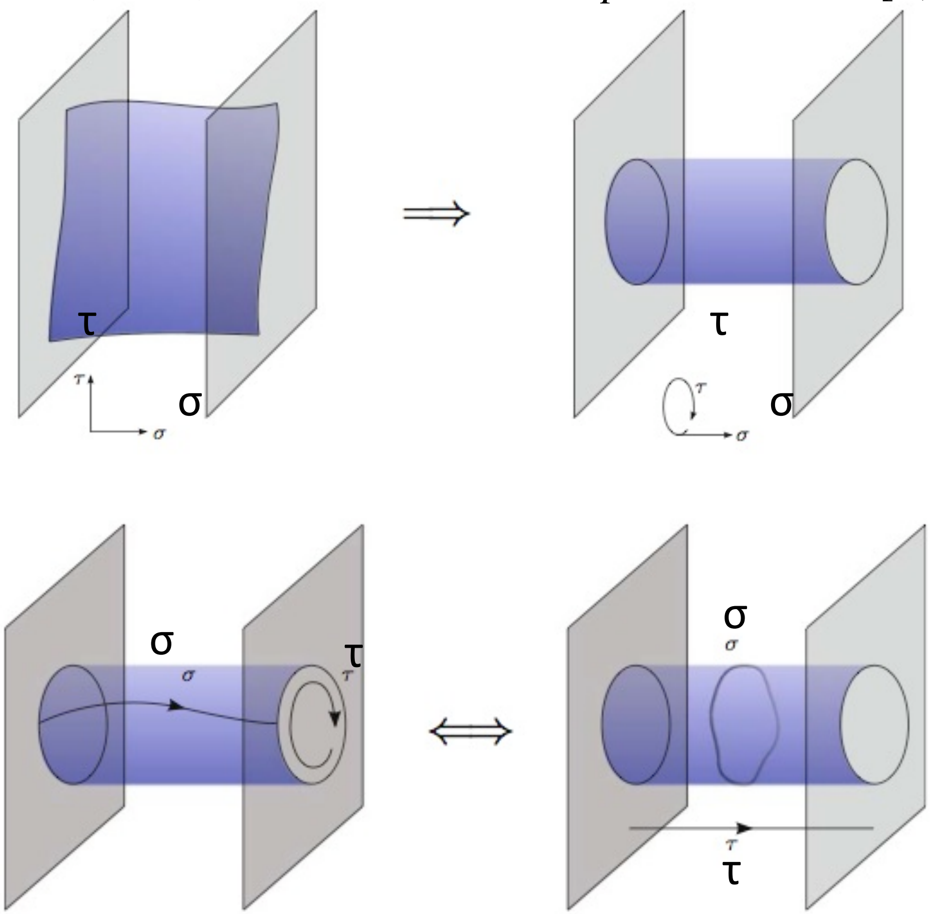
- For both Neumann and Dirichlet boundary conditions, we have

$$\boxed{L_n - \bar{L}_n = 0} \quad \Rightarrow \quad T(z) = \bar{T}(\bar{z})$$

It means the central charge in holomorphic and anti-holomorphic theories have to be equal.

1.2.1 A Glimpse of Partition Function

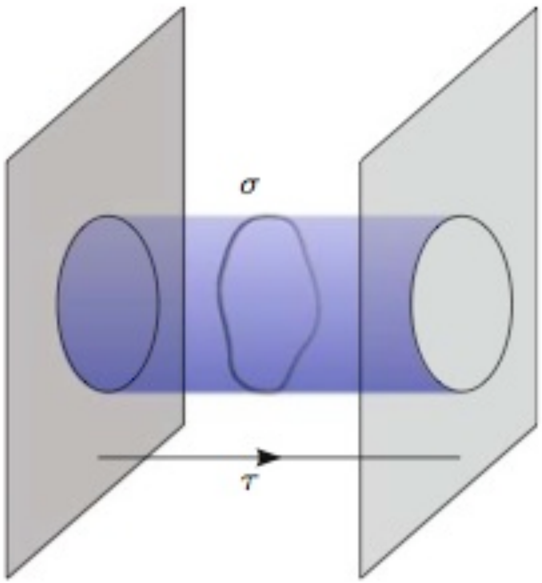
$\tau \in (-\infty, \infty) \quad \sigma \in [0, \pi]$ $\qquad$ $\tau \;\; periodic \quad \sigma \in [0, \pi]$



$$(\sigma, \tau)_{open} \leftrightarrow (\tau, \sigma)_{closed}$$

- One loop partition function for CFTs defined on a **torus**
- For BCFT the topology of a **cylinder** instead of a torus is yielded
- One-loop partition function in BCFT is equivalent to tree-level amplitude in CFT: ***Loop-channel – Tree-channel Equivalence***
- Notation: $Z_{loop} \to Z$ $\quad$ $Z_{tree} \to \tilde{Z}$

1.2.2 Boundary States



- Focus on Closed sector (Tree-Channel)
- Action formalism is not as general as Hilbert space formalism.
- Consider a Hilbert space of some theory. In the presence of a boundary, there are some particular states satisfying the boundary conditions, we call them ***Boundary States***

$$\partial_\tau X_{closed}\big|_{\tau=0} \left|B_N\right\rangle = 0 \quad \text{Neumann condition}$$

$$\partial_\sigma X_{closed}\big|_{\tau=0} \left|B_D\right\rangle = 0 \quad \text{Dirichlet condition}$$

1.2.3 Boundary States and the Laurent Modes

- Boundary Conditions into the picture of Boundary States:

$\partial_\tau X_{closed}\big\vert_{\tau=0}\left\vert B_N\right\rangle = 0$	Neumann condition
$\partial_\sigma X_{closed}\big\vert_{\tau=0}\left\vert B_D\right\rangle = 0$	Dirichlet condition

- We can also express them in terms of the Laurent modes:

$$i\cdot\partial_\tau X_{closed}\Big\vert_{\tau=0} = \sum_{n\in Z}(j_n e^{-in\sigma} + \bar{j}_n e^{+in\sigma})$$

$$\partial_\sigma X_{closed}\Big\vert_{\tau=0} = \sum_{n\in Z}(j_n e^{-in\sigma} - \bar{j}_n e^{+in\sigma})$$

Relabel n-> -n

$$j_n + \bar{j}_{-n}\left\vert B_N\right\rangle = 0, \quad (\pi_0\left\vert B_N\right\rangle = 0) \quad N.C.$$

$$j_n - \bar{j}_{-n}\left\vert B_D\right\rangle = 0 \quad D.C.$$

Gluing Conditions.

1.2.4 Solutions to gluing conditions

- The solution for the gluing conditions for the example of free boson

$$|B_N\rangle = \frac{1}{N_N}\exp(-\sum_{k=1}^{\infty}\frac{1}{k}j_{-k}\bar{j}_{-k})|0\rangle \qquad N.C.$$

$$|B_D\rangle = \frac{1}{N_D}\exp(+\sum_{k=1}^{\infty}\frac{1}{k}j_{-k}\bar{j}_{-k})|0\rangle \qquad D.C.$$

- It has the following structure

After some calculation on the blackboard……

$$|B\rangle = \frac{1}{N}\sum_{\vec{m}}|\vec{m}\rangle \otimes |U\vec{m}\rangle$$

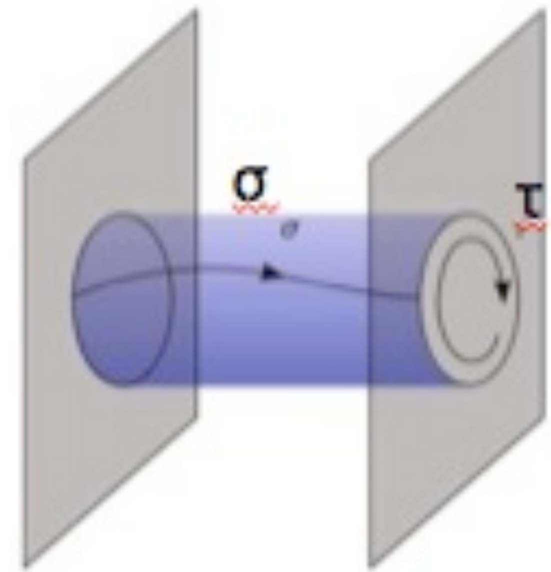
1.3.1 Loop-Channel Partition Function (Open Sector)

- the loop partition function for free boson on cylinder is given by

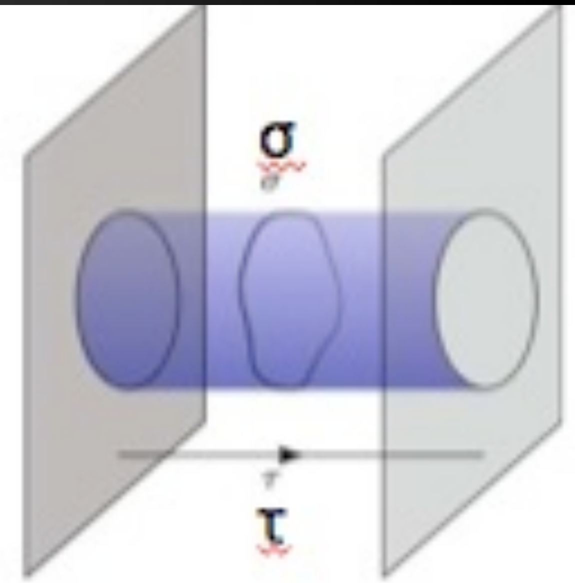
$$Z^C(t) = Tr_{H_B}(q^{L_0 - c/24})$$

- For example, the Neumann-Neumann boundary condition

$$Z^{C(N,N)}_{bos.}(t) = \frac{1}{2\sqrt{t}}\frac{1}{\eta(it)}$$



1.3.2 Tree-Channel Partition Function (Closed Sector)



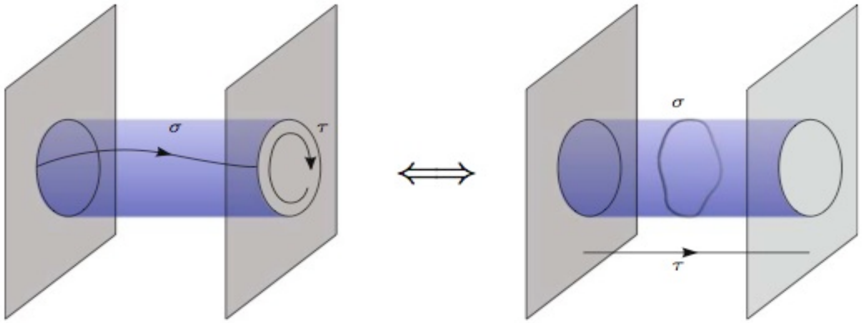
- The tree-level amplitude is given by the overlap:

$$\tilde{Z}^{C}(l) = \left\langle \Theta B \left| e^{-2\pi l (L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24})} \right| B \right\rangle$$

- Finally, the partition function for the closed sector is calculated to be (for the example of Neumann-Neumann boundary condition)

$$\tilde{Z}^{C(N,N)}_{bos.}(l) = \frac{1}{N_N^2} \frac{1}{\eta(2il)}$$

1.3.3 Loop-Channel – Tree-Channel Equivalence



the open and closed sector are related by

$$(\sigma,\tau)_{open} \leftrightarrow (\tau,\sigma)_{closed}$$

Same cylinder: $\boxed{t = \frac{1}{2l}}$

$$\boxed{|B\rangle = \frac{1}{N}\sum_{\vec{m}} |\vec{m}\rangle \otimes |U\vec{\bar{m}}\rangle}$$

Fix the normalization constants.

$$Z_{bos.}^{C(N,N)}(t) = \frac{1}{2\sqrt{t}}\frac{1}{\eta(it)} \quad \rightarrow$$

$$\sqrt{\frac{l}{2}}\frac{1}{\eta(-\frac{1}{2il})}\frac{1}{2\eta(2il)} = \frac{N_N^2}{2}\tilde{Z}_{bos.}^{C(N,N)}(l)$$

$\Rightarrow$ $\boxed{N_N = \sqrt{2}}$

Similarly, $\boxed{N_D = 1}$

2.1 Boundary states for Rational CFT

- As for the free boson model, a boundary state in RCFT is required to satisfy the following gluing conditions:

$$\left(L_n - \bar{L}_{-n}\right)\left|B\right\rangle = 0 \quad \text{Conformal symmetry}$$

- **Ishibashi States**: $\left|\beta_i\right\rangle\rangle = \sum_{\vec{m}} \left|\phi_i, \vec{m}\right\rangle \otimes U\left|\bar{\phi}_i, \vec{\bar{m}}\right\rangle$

  satisfy the gluing condition

- Question: Are Ishibashi states really our boundary state?

2.1 Boundary states for Rational CFT

- Consider the following overlap of the Ishibashi states:

$$\langle\langle \beta_j | \exp\left[-2\pi l\left(L_0 + \bar{L}_0 - (c+\bar{c})/24\right)\right] | \beta_i \rangle\rangle$$
$$= \delta_{ij^+} \langle\langle \beta_i | \exp\left[-2\pi i(2il)\left(L_0 - c/24\right)\right] | \beta_i \rangle\rangle$$
$$= \delta_{ij^+} Tr_{H_i}(q^{L_0 - c/24}) \quad = \delta_{ij^+} \chi_i(2il)$$

- S-transform of *χ(2il)* does not give non-negative integer coefficients as required by Verline formula: $Z_{\alpha\beta}(t) = \sum_j n^j_{\alpha\beta} \chi_j(it)$

- A true boundary States $|B_\alpha\rangle = \sum_i B^i_\alpha |\beta_i\rangle\rangle$

- The complex coefficient are constrained by the ***Cardy Condition.***

2.2 Cardy condition

- Now, the cylinder amplitude between two boundary states can be expressed as

$$\begin{aligned}\tilde{Z}_{\alpha\beta}(l) &= \left\langle \Theta B_\alpha \middle| \exp\left[-2\pi l\left(L_0 + \bar{L}_0 - (c+\bar{c})/24\right)\right] \middle| B_\beta \right\rangle \\ &= \sum_{i,j} B_\alpha^j B_\beta^i \left\langle\left\langle \beta_{j^+} \middle| \exp\left[-2\pi l\left(L_0 + \bar{L}_0 - (c+\bar{c})/24\right)\right] \middle| \beta_i \right\rangle\right\rangle \\ &= \sum_i B_\alpha^i B_\beta^i \chi_i(2il)\end{aligned}$$

2.2 Cardy condition

- **Loop-channel – Tree-channel Equivalence**
closed sector cylinder diagram is transformed to the following in the open sector:

$$\tilde{Z}_{\alpha\beta}(l) \xrightarrow{l\to 1/2t} = \tilde{Z}_{\alpha\beta}\left(\frac{1}{2t}\right) = \sum_{i,j} B_\alpha^i B_\beta^i S_{ij} \chi_j(it) \quad =? \quad Z_{\alpha\beta}(t)$$

$$Z_{\alpha\beta}(t) = \sum_j n_{\alpha\beta}^j \chi_j(it)$$

- **Cardy Condition** ***ensures*** **Loop-channel – Tree-channel Equivalence**

$$\tilde{Z}_{\alpha\beta}(l) \xrightarrow{l\to 1/2t} Z_{\alpha\beta}(t)$$

$$n_{\alpha\beta}^j = \sum_j B_\alpha^i B_\beta^i S_{ij} \qquad \in Z_0^+$$

That is, for all pairs of the boundary states in a RCFT, such combinations have to be non-negative integers.

3.1 Ground-State Degeneracy

- Consider *log Z*, the effect of boundary leads to one other term *log g*, i.e.

$$\log Z_{\alpha\beta} \to \log Z_{\alpha\beta} + \log g \qquad Z_{\alpha\beta} \to g \cdot Z_{\alpha\beta}$$

- g is then called ***Ground State Degeneracy***

$$g = \langle 0|\alpha\rangle\langle\beta|0\rangle \qquad g = g_\alpha g_\beta, \quad g_\alpha = \langle 0|\alpha\rangle$$

- Example of free boson:

$$Z(t) = \frac{1}{\eta(it)} \quad \Rightarrow \quad g = \frac{1}{2\sqrt{t}}$$

$$Z_{bos.}^{C(N,N)}(t) = \frac{1}{2\sqrt{t}}\frac{1}{\eta(it)}$$

- g is **universal**: does not depend on length or mass or any energy scale, only depends on boundary.

3.1 Ground-State Degeneracy

- Once again, consider the cylinder partition function with Verlinde formula:

$$Z^0_{\alpha\beta} = Tr(e^{\frac{\pi}{l}(L_0 - c/24)}) \quad = \sum_i n^i_{\alpha\beta}\chi_i$$

- The infinite limit of *l* could be expressed as modular transformation for the character:

$$\chi_i = \sum_j S_{ij}\chi_j$$

- at infinite limit of *l*, only ground state contributes to the expression, $\sum_j S_{ij}\chi_j$

thus $Z_{\alpha\beta} \rightarrow Tr(e^{\frac{\pi}{l}(L_0 - c/24)})\sum_i n^i_{\alpha\beta}S_{i0} \rightarrow g \cdot Z^0_{\alpha\beta}$ $\quad$ *with* $\boxed{g = g_\alpha g_\beta = \sum_i n^i_{\alpha\beta}S_{i0}}$

- Interpretation: the consistency of this formula with $g_\alpha = \langle 0|\alpha\rangle$ puts certain constrain on the possible boundary states and the coefficients *n*. This constraint is ***Cardy Condition.***

Topics

- Part I: Boundary Conditions & Boundary States
    - 1.1 Conformal Invariance & Boundary Conditions
    - 1.2 Boundary States & Gluing Condition
    - 1.3 Partition function & Loop-channel – Tree channel Equivalence
- Part II: Cardy condition
- Part III: g-function

Thank you for your attention.