Gauge/Gravity Duality

The AdS_5/CFT_4 Correspondence

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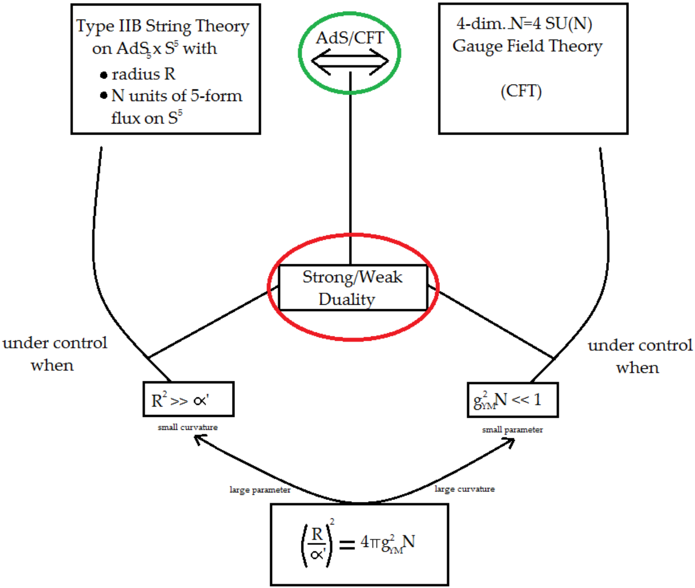

Type IIB String Theory on AdS5 x S5 with
radius R
N units of 5-form flux on S5
AdS/CFT
4-dim. N=4 SU(N) Gauge Field Theory
(CFT)
Strong/Weak Duality
under control when
under control when
R2 >> α'
small curvature
g2YM N << 1
small parameter
large parameter
large curvature
(R/α')2 = 4πg2YM N

Presentation Layout

Part 0: Large N Gauge Theories as String Theories

1. The 't Hooft limit
2. Perturbative diagram expansion
3. Link with Strings

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Part I: D3-branes: Gauge Theories and Gravity Solutions

1. D3-branes and gauge theories
2. Global and Gauge Symmetries Groups
3. Black p-branes as classical SuGra solutions

Presentation Layout

Part II: AdS_5/CFT_4 Correspondence

1. Low Energy Limits
2. AdS/CFT Conjecture, Duality
3. Formulation
4. Holography

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Part III: AdS_5/CFT_4 Correlation Functions

1. General method
2. 2-point functions and bulk-to.boundary propagators
3. n-point functions and boundary-to-boundary propagators

Part 0

Large N Gauge Theories as String Theories

or How special limits of special theories take you to a special place

The 't Hooft limit

Consider $U(N)$ Yang-Mills theory, coupling constant g_{YM}
$\rightarrow$ Gauge fields in the adjoint representation of $U(N)$ A_μ^a with field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_{YM} \left[A_\mu, A_\nu\right]$$

$\rightarrow$ Lagrangian density

$$\mathcal{L} = \frac{1}{g_{YM}^2} \left(Tr(F_{\mu\nu}^2) + \mathcal{L}_{matter} \right)$$

t'Hooft parameter: $\lambda \equiv g_{YM}^2 N \Longrightarrow \mathcal{L} \sim N/\lambda$

t'Hoot limit: $N \rightarrow \infty$ and λ fixed

Represent the adjoint gauge field $A_\mu = A_\mu^a T^a$ as a direct product of fundamental and anti-fundamental fields:

$\rightarrow$ ${A^i}_j$, $N \times N$ hermitian matrices

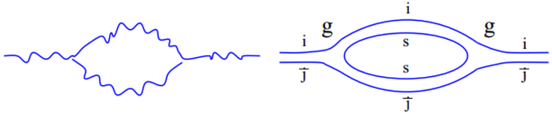
$\rightarrow$ $U(N)$ propagator: $< {A^j}_i {A^l}_k > \propto \; {\delta^l}_i {\delta^j}_k$

$\rightarrow$ Feynman diagrams: network of double lines

Vacuum diagrams: compact closed oriented surfaces.

Planar Perturbative Expansion

Example: gluon self energy



$\rightarrow$ free index s taking N different values: $graph \;\propto O(g_{YM}^2 N)$, finite in t'Hooft limit

$$\mathcal{L} = \frac{N}{\lambda}\left(Tr(F_{\mu\nu}^2) + \mathcal{L}_{matter}\right)$$

Feynman rules:

- N/λ for each vertex (V)
- λ/N for each propagator (E, edge)
- N for each loop (F, face)

$\Longrightarrow N^{V+F-E}\lambda^{E-V} = N^{\chi}\lambda^{E-V}$ for each vacuum bubble graph

Perturbative Expansion

For closed oriented surfaces, $\chi = 2 - 2g$

$\rightarrow$ perturbative expansion

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) = \sum_{g=0}^{\infty} \left(\frac{1}{N}\right)^{2g} N^2 f_g(\lambda)$$

$\rightarrow$ large N limit: dominated by $\boxed{\text{maximal } \chi/\text{minimal } g}$, sphere topology

$\rightarrow$ correspond to the perturbative theory of closed oriented strings

In general, $S \rightarrow S + N \sum_j J_j G_j$ and

$$\langle \Pi_{j=1}^{n} G_j \rangle = (iN)^{-n} \left[\frac{\delta^n W}{\Pi_{j=1}^{n} \delta J_j} \right]_{J_j=0} \propto N^{2-n}$$

$\rightarrow$ $1/N$ as a coupling constant g_s

Part I

D3-branes
Gauge Theories and Gravity Solutions

or How to describe something in two amazing different ways

Gauge Theory on D3-branes

System: N parallel D3-branes in type IIB string theory in 10d

$\rightarrow$ low energy effective theory: U(N) gauge theory in (3+1)-dimensions with 16 supercharges.

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- λ^a_α, $\alpha = 1, 2$, $a = 1, ..., 4$ left Weyl fermionic fields
- X^i, $i = 1, ..., 6$ real scalar fields - $SO(6) \sim SU(4)$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu]$, A_μ gauge fields with field strength $F_{\mu\nu}$
- $D_\mu\lambda = \partial\lambda + i\,[A_\mu, \lambda]$, a covariant derivative.

Formulation of $\mathcal{N} = 4$ SYM

The Lagrangian is completely settled by supersymmetry:

$$\mathcal{L} = Tr\left(-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu X^i D^\mu X_i + \sum_{a,b,i} gC_i^{ab}\lambda_a\left[X^i,\lambda_b\right] + \sum_{a,b,i} g\bar{C}_{iab}\bar{\lambda}^a\left[X^i,\bar{\lambda}_b\right] + \frac{g^2}{2}\sum_{i,j}\left[X^i,X^j\right]^2\right)$$

$$[\lambda^a] = \frac{3}{2} \qquad \left[X^i\right] = 1 \qquad [A_\mu] = 1, \qquad [g_{YM}] = 0$$

$\Longrightarrow$ renormalisable theory
Quantum level: $\mathcal{N} = 4\ SYM$ exhibits no UV divergences! Also, it preserves all its symmetries!

Symmetries of $\mathcal{N} = 4$ SYM

- R-symmetry $\boxed{SO(6) \sim SU(4)}$, with generators $T^A,\ A = 1, ..., 15$
- scale invariance + Poincaré invariance
  $\longrightarrow$ conformal symmetry in 4d $\boxed{SO(2,4)}$ with generators $P_\mu, L_{\mu\nu}, D, K_\mu$
- even more, $\mathcal{N} = 4$ Poincaré Supersymmetry + conformal invariance
  $\longrightarrow$ superconformal symmetry $SU(2,2|4)$, with superalgebra

$$\left(\begin{array}{c|c} P_\mu, K_\mu, L_{\mu\nu}, D & Q^a_\alpha, \bar{S}^a_{\dot\alpha} \\ \hline \bar{Q}_{\dot\alpha a}, S_{\alpha a} & T^A \end{array}\right)$$

D3-branes: Source of Gravity

Now for something completely different.

D3-branes are **massive charged objects** in a gravity theory: string theory!

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What happens when taking the same low energy limit, and describe it with string theory?

$\sim$ Replace D-branes by a non-trivial geometry, as in everyday life physics!

Black p-Branes

Again, type IIB string theory in 10-d.

Low energy effective action

$$S = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} \left(e^{-2\phi}(R + 4(\nabla\phi)^2) - \frac{2}{(5)!} F_5^2 \right)$$

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$\rightarrow$ Look for solution as 3-d electric source of charge N for A_4.

$$N = \int_{S^5} *F_5$$

Require desirable symmetries: $ISO(1,3)$ in 3-d along D3-brane, and spherical symmetry $SO(6)$ in 6 transversal directions.

Classical SuGra Solution

In 4-d Gravity with point-like object: Reissner-Nördstrom black hole solution

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Extremal Solution:

$$\boxed{ds^2 = \frac{1}{\sqrt{H(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(r)}(dr^2 + r^2 d\Omega_5^2)}$$

with $H(r) = 1 + \frac{R^4}{r^4}$ $\qquad R^4 = 4\pi g_s \alpha'^2 N$

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This description is appropriate at low curvature of the 3-brane geometry:

$$R \gg l_s \implies 1 \ll g_s N \ll N$$

Near Horizon Limit

Duality coming from this double-description:

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N D3-branes can consistently be described in a low energy limit by

- a U(N) gauge theory on the world-volume
- a string theory in a background which near the horizon is $AdS_5 \times S^5$ with radius $R^4 = 4\pi g_s \alpha'^2 N$

Part II

$\mathrm{AdS}_5/\mathrm{CFT}_4$ Correspondence

or How to get to the point

Low Energy Limit Action

N parallel D3-branes

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$$\Longrightarrow \quad S = S_{bulk} + S_{brane} + S_{int}$$

Take $\alpha' \rightarrow 0$, g_s, N fixed:
$\rightarrow$ S_{int} vanishes
$\rightarrow$ S_{bulk} becomes free gravity
$\rightarrow$ S_{brane} becomes pure $\mathcal{N} = 4\ U(N)\ SYM$

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$\rightarrow$ states of any mass close to the horizon $r = 0$ will survive

$\rightarrow$ massless states away from the horizon also survive

$\rightarrow$ these two sets of states are decoupled from each other

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Remember: near horizon region is $AdS_5 \times S^5$

Conclusion:
In both approaches, we get two decoupled theories:

10d free gravity in the bulk	10d free gravity in the bulk
4d gauge theory in the branes	near horizon full theory

AdS/CFT Conjecture

Identify the two approaches and be forced to make the conjecture:

$\mathcal{N} = 4\ U(N)\ SYM$ in flat 3+1 dimensions

is "equivalent" to

Type IIB superstring theory on $AdS_5 \times S^5$

Symmetries

String theory

- AdS_5 has isometry group $SO(2,4)$
- S^5 has rotational symmetry $SO(6)$

$\mathcal{N} = 4$ SYM

- 4d conformal symmetry $\simeq SO(2,4)$
- it also has global symmetry $SO(6)$

In fact, the whole supersymmetric group match on both sides

$\rightarrow$ both sides of the conjecture have the same spacetime symmetries!

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$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1 \qquad \text{Perturbative FT Regime}$$

The classical gravity description can be trusted when

$$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1 \qquad \text{Gravity Regime}$$

$\Longrightarrow$ perfectly incompatible: $\boxed{\text{duality}}$

Boundary Considerations

Problem: how to link AdS_5 fields to CFT_4 operators?

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Fact: AdS space has a boundary; any Field Theory on AdS needs boundary conditions on its fields to be solvable.
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Solution: for an operator O in CFT$_4$, add to the Lagrangian

$$\int d^4x \; \phi_0(\vec{x})O(\vec{x})$$

where a field $\phi(\vec{x}, z)$ in AdS_5 has boundary condition

$$\phi(\vec{x}, z)\big|_{z=0} = \phi_0(\vec{x})$$

Formulation

$\Longrightarrow$ value of $\phi(\vec{x}, z)$ on the boundary act as source to operator O.

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So, naturally propose:

$$\langle e^{\int d^4x \;\phi_0(\vec{x})O(\vec{x})}\rangle_{CFT} \stackrel{\text{AdS/CFT}}{=} Z_{AdS}^{string}\left[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})\right]$$

which matches

the generating functional of O-correlation functions
with
the full partition function of string theory (in AdS) with boundary condition.

This is the way correlation functions may be computed.

Note: The coupling requires field ϕ and operator O to have the same quantum numbers of the theories' symmetry group!

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6. Despair

Small detour on Holography.

The AdS/CFT correspondence is an explicit realisation of holography.

By matching every observable on AdS_5 to every observable on CFT_4, we apparently reduce the dimension of our theory by 1 without any problem.

Where do the mismatching degrees of freedom go?

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Solution:

Holographic principle: in a full quantum gravity description this is not a problem, it is even required.

Entropy Bound

Black-hole entropy: $S_{BH} = \frac{A}{4}$

$\rightarrow$ generalised 2nd law of thermodynamics: $dS_{BH+matter} \geq 0$

This law requires an entropy bound for matter.

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This law requires an entropy bound for matter.

Spherical entropy bound:

$$\boxed{S_{matter} \leq \frac{A}{4G_N}}$$

Entropy and degrees of freedom

In a quantum system, e^S is the number of independent quantum states compatible with certain macroscopic parameters, e.g. $E, T, V, ...$

This is also the number of degrees of freedom of the quantum theory where the system is described.

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This is also the number of degrees of freedom of the quantum theory where the system is described.
More precisely,

$$N \equiv \#dof = log\ dim(\mathcal{H})$$
$$dim(\mathcal{H}) = e^S$$

For a QFT: $N = \infty$
For a regularised QFT (e.g. quantum gravity): $N \sim V$
Contradiction: $S \leq A/4 \Longrightarrow N \sim A$

Holographic Principle

We aknowledge the previous result as a law of nature and generalise the concept.

Holographic Principle:

"In a quantum theory of gravity all physics within some volume can be described in terms of some theory on the boundary which has less than one d.o.f. per Planck area"

$\rightarrow$ so that it satisfies the required bound.

In a nutshell, our theories have a redundacy in the form of useless d.o.f. ; this redundancy can be solved by considering some "dual theories" on the boundary of the space our theories live in.

AdS/CFT and Holography

The AdS/CFT correspondence explicitly realises this principle.

How: count the degrees of freedom and compare with the area.

Problem: CFT has infinitely many d.o.f.; boundary of AdS has infinite area...
$\rightarrow$ regularise infinities and compare them.

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Problem: CFT has infinitely many d.o.f.; boundary of AdS has infinite area...
$\rightarrow$ regularise infinities and compare them.

- Regularise the boundary: discrete cells of size δ, 1 d.o.f. per cell
- # cells $\sim \delta^{-3}$
- # d.o.f. in U(N) with this UV cut-off: $S \sim N^2 \delta^{-3}$
- $\Longrightarrow$ we have a IR cut-off at $z \sim \delta$ on the bulk

Area in AdS_5 of the surface $z \sim \delta$:

$$A \sim \frac{R^3}{\delta^3}$$

Given that $G_5 \sim N^{-2} R^3$,

$$\implies S \sim A/G_5$$

The holographic bound is thus saturated by the AdS/CFT correspondence.

AdS/CFT is a realisation of the Holographic Principle.

Part III: AdS_5/CFT_4 Correlation Functions

or How to find old CFT friends

Correlation Functions

So,

$$Z_{CFT}\left[\phi_0\right] \equiv \langle e^{\int d^4x \;\; \phi_0(\vec{x})O(\vec{x})} \rangle_{CFT} \; = \; Z_{AdS}^{string}\left[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\right]$$

we meet again.

So,

$$Z_{CFT}\left[\phi_0\right] \equiv \langle e^{\int d^4x \ \phi_0(\vec{x}) O(\vec{x})} \rangle_{CFT} \ = \ Z_{AdS}^{string}\left[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})\right]$$

we meet again.

Computing O-correlation functions:

$$\langle O...O \rangle = \left. \frac{\delta^n Z_{CFT}\left[\phi_0\right]}{\delta \phi_0^n} \right|_{\phi_0 = 0}$$

$\rightarrow$ $\phi(\vec{x}, z)$ solves e.o.m. derived from S_{AdS_5} (on-shell), with given boundary conditions
$\rightarrow$ the extension from ϕ_0 to ϕ is unique

Massless Scalar Field 2-pt Function

Consider

$$S_{AdS}\left[\phi\right] = \frac{1}{2}\int d^{d+1}x\sqrt{g}\,\left(\partial\phi\right)^2$$

Idea

- solve ϕ in terms of ϕ_0 with regularised boundary condition
- evaluate $S\left[\phi\right]$ on $\phi \rightarrow \quad S_{AdS}\left[\phi_0\right]$
- take functional derivatives w.r.t. ϕ_0

$\rightarrow \quad \phi$ is on-shell: integrate S_{AdS} by parts. The term giving the e.o.m. is zero, the remaining regularised boundary term is

$$S_{AdS}\left[\phi\right] = \lim_{\epsilon\rightarrow 0}\frac{1}{2}\int_{T_\epsilon} d^dx\sqrt{h}\;\phi\;\partial_n\phi$$

Scalar Field 2-pt. Function

$$\rightarrow \quad S_{AdS}\left[\phi_0\right] = \tfrac{cd}{2} \int \; d\vec{x} \; d\vec{x}' \; \tfrac{\phi_0(\vec{x})\phi_0(\vec{x}')}{|\vec{x}-\vec{x}'|^{2d}}$$

2-pt function in a CFT of operator O with conformal dimension

$$\boxed{\Delta = d}$$

Also:

$$\langle O \rangle = \left. \frac{\delta S_{AdS}}{\delta \phi_0} \right|_{\phi_0 = 0} = 0$$

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Massive case: more complicated. Result:

$$\boxed{\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2R^2}\right)}$$

General Conception

Massive interacting scalar fields:

$$S_{AdS} = \int d^5x\sqrt{g}\left(\frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}m_i^2\phi_i^2 + \sum_{k=3}^{m}\lambda_{i_1\ldots i_k}\phi_{i_1}\ldots\phi_{i_k}\right)$$

Typical e.o.m.: $(\Box + m^2)\phi = \lambda\phi^n$

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Perturbative expansion on λ:

$$\text{at } \lambda^0 : \quad \phi^{zero}(x,z) = \int K(x-x',z)\ \phi_0(x')$$

$$\text{at } \lambda^1 : \quad \phi^{one}(x,z) = \int G(x-x',z-z')(\phi^{zero}(x',z'))^n$$

where $G(x-x',z-z')$ is a bulk-to-bulk Green's function.

THANK YOU FOR YOUR ATTENTION!