The Quantum Marginal Problem

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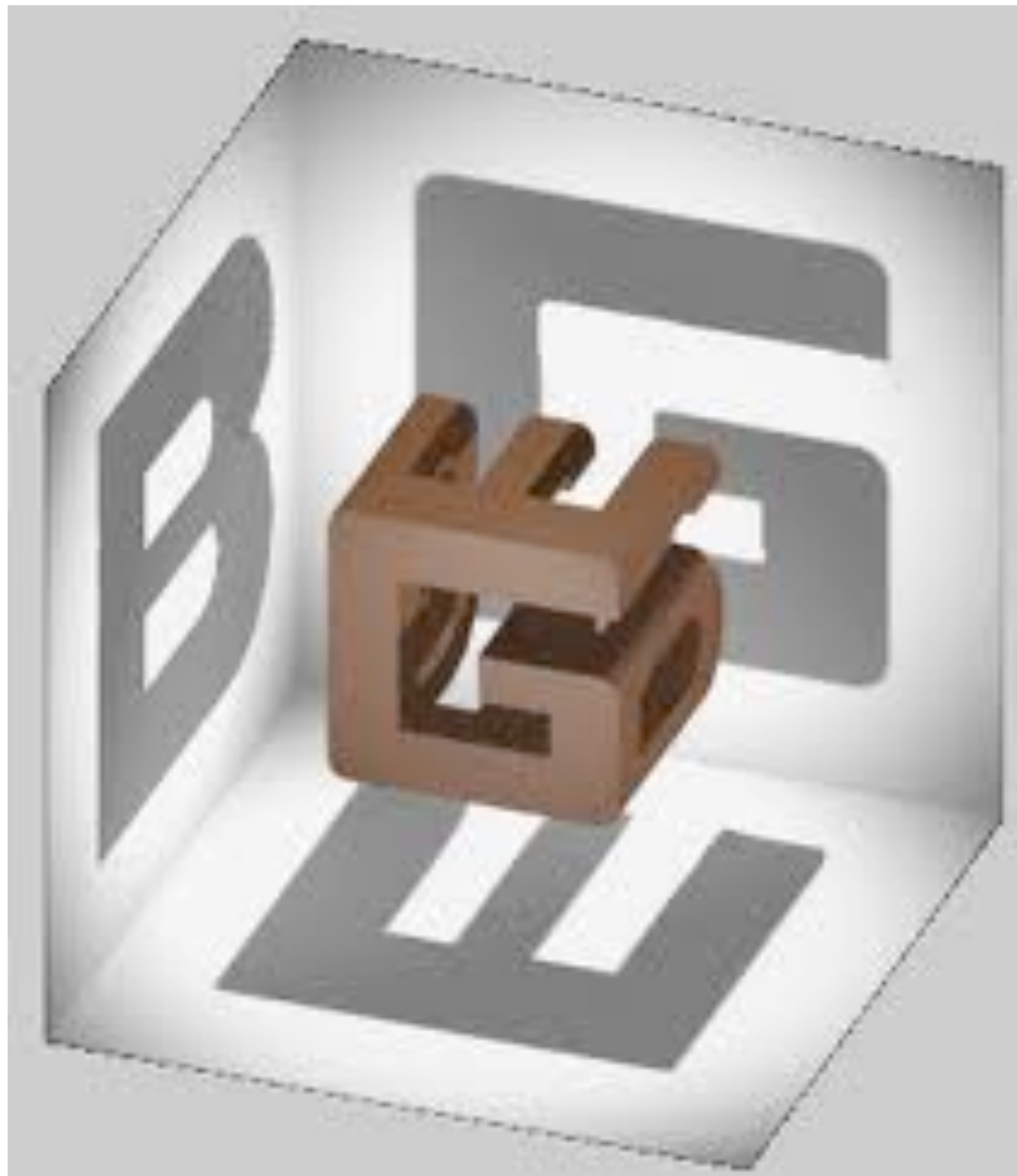
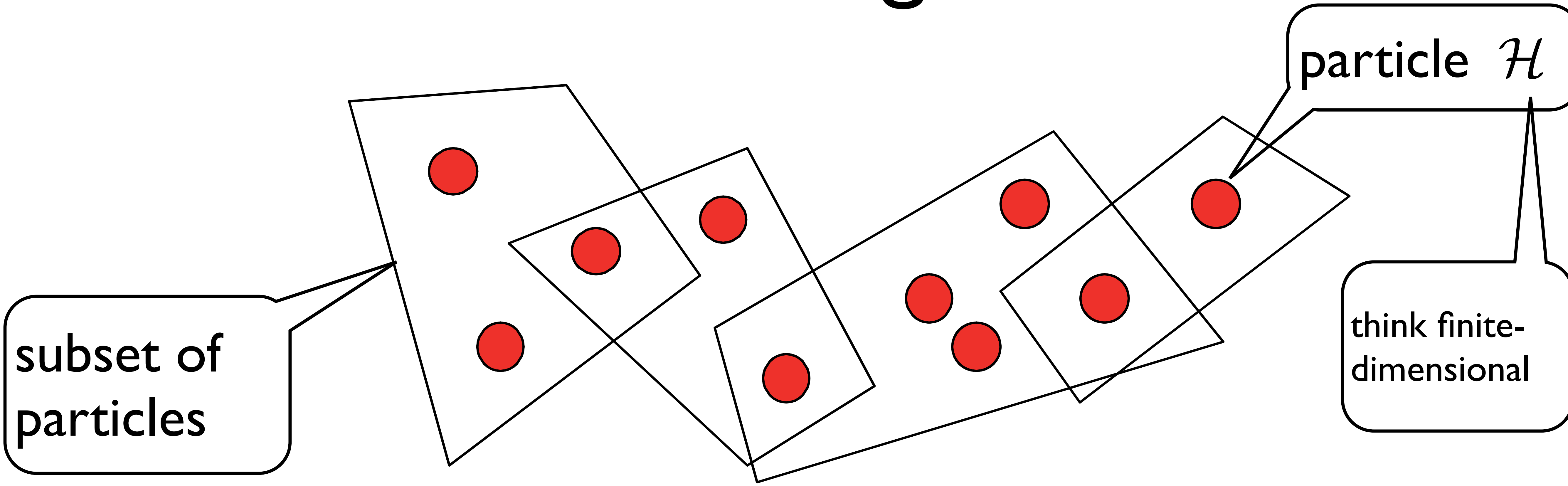
Gödel, Escher & Bach



For which triples of letters is this possible?

The Quantum Marginal Problem

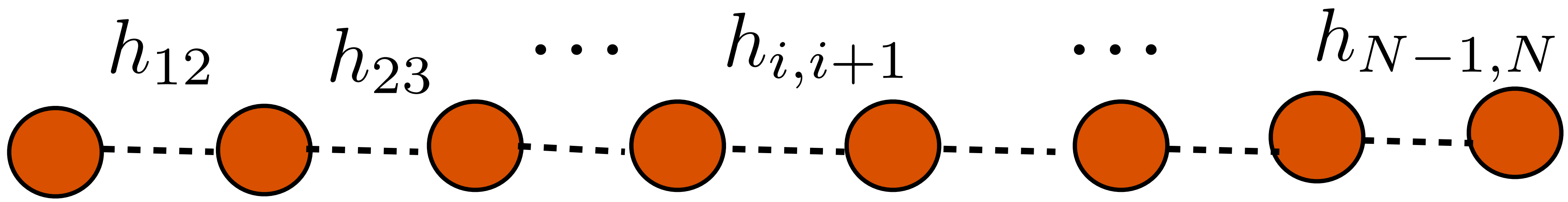


Fix subsets of the particles $S_i \subseteq \{1, \ldots, N\}$

For each subset, given a density matrix ρ_{S_i}

Are these compatible?

$$\exists ?\ \rho_{\{1,\ldots,N\}} :$$
$$tr_{\bar{S}_i} \rho_{\{1,\ldots,N\}} = \rho_{S_i}$$



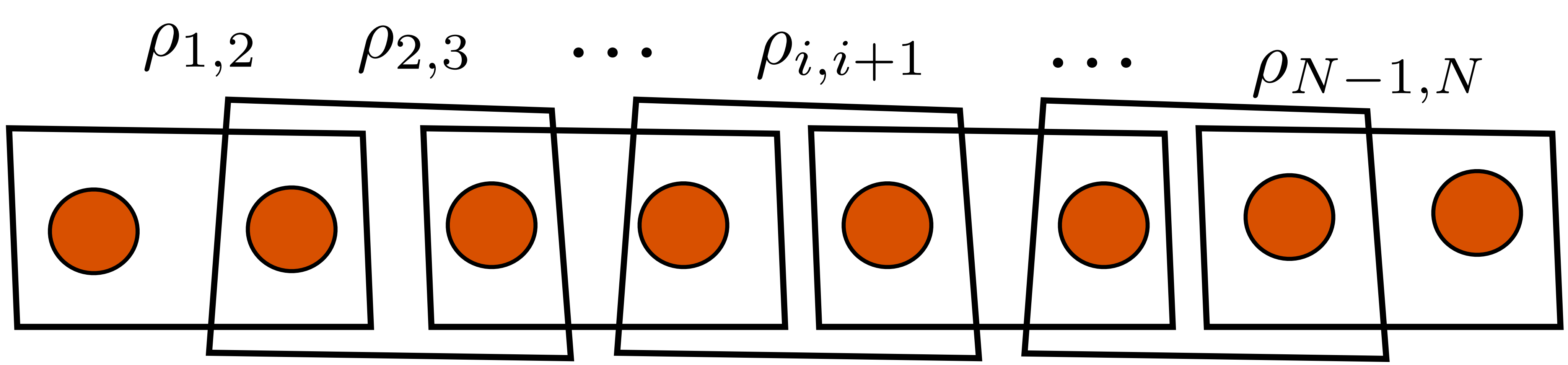
$$min_\rho tr\rho H = min_\rho \sum_i tr\rho \; id \otimes \cdots id \otimes h_{i,i+1} \otimes id \otimes \cdots \otimes id$$

$$= min_\rho \sum_i tr\rho_{i,i+1} h_{i,i+1} = min_{\rho_{i,i+1}} \sum_i tr\rho_{i,i+1} h_{i,i+1}$$

exp(N) variables

poly(N) variables

compatible!



The Quantum Marginal Problem

- studied since beginnings of quantum theory
- computionally difficult
  QMA-complete (Liu, 2006) ⇒NP-hard
- fermionic version
  quantum chemistry
  QMA-complete (Liu, Ch.& Verstraete, 2007)
- partial understanding
  Pauli principle
  Entropy inequalities
  (Lieb& Ruskai 1973, Pippenger 2003)

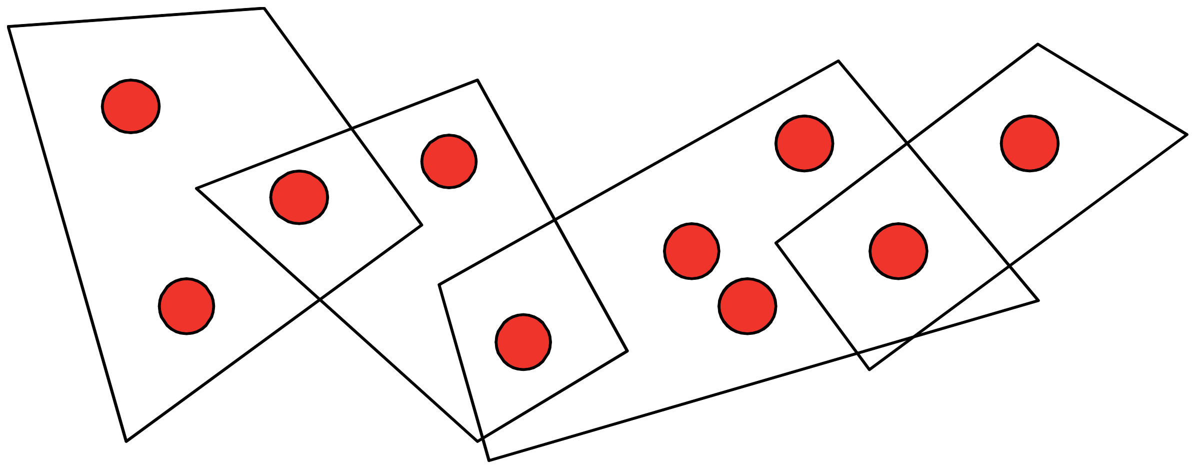
currently in quantum information and computation

occupation numbers

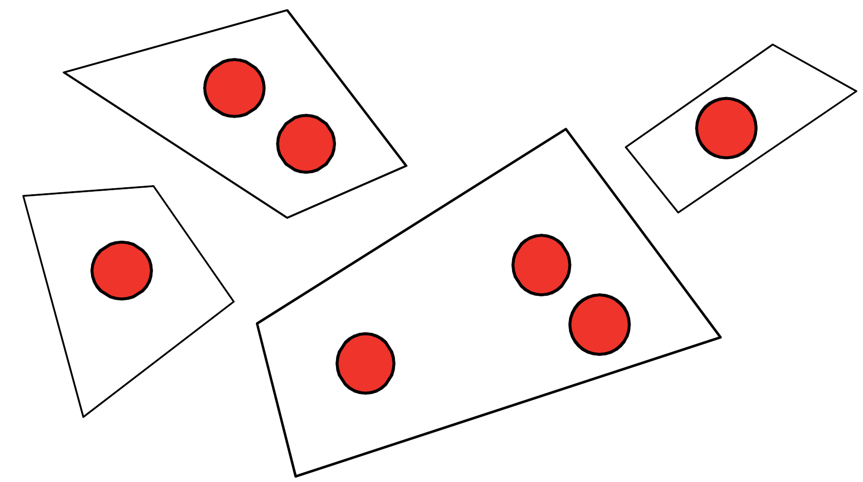
$$\lambda_i \leq 1$$

$$S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_2) + S(\rho_{123})$$

v. Neumann entropy



Collection of subsets of a set of particles (overlapping)



Collection of subsets of a set of particles (non-overlapping)

Fix subsets of the particles $S_i \cap S_j = \emptyset$

what if pure?

For each subset, given a density matrix ρ_{S_i}

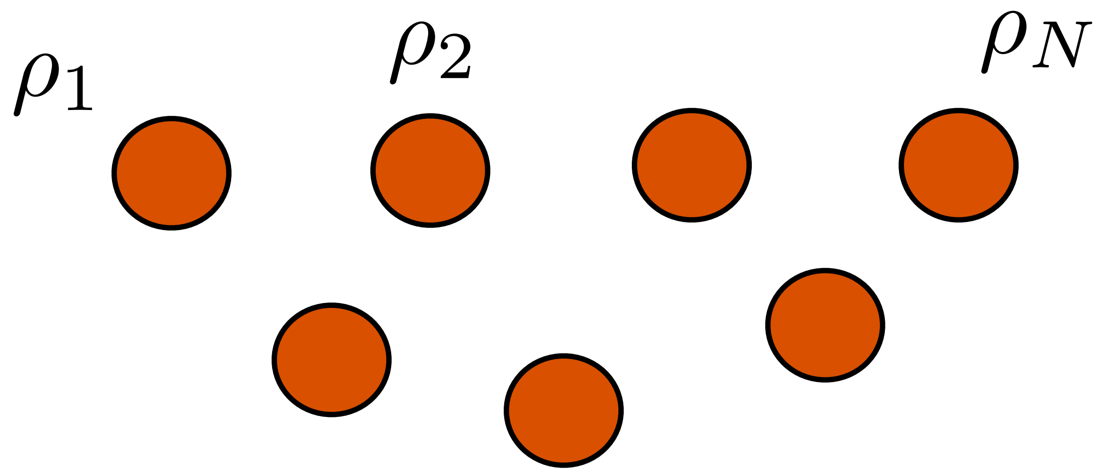
Are these compatible? $\exists ?\ \rho_{\{1,\ldots,N\}} :$

$$tr_{\bar{S}_i} \rho_{\{1,\ldots,N\}} = \rho_{S_i}$$

Yes!

$$\rho_{1,\ldots,N} = \otimes_i \rho_{S_i}$$

One-Body Quantum Marginal Problem



If ρ_i compatible: $tr_{\bar{i}}|\psi\rangle\langle\psi| = \rho_i$

Then $\tilde{\rho}_i := u_i \rho_i u_i^\dagger$ compatible: $tr_{\bar{i}}|\tilde{\psi}\rangle\langle\tilde{\psi}| = \tilde{\rho}_i$

$|\tilde{\psi}\rangle := u_1 \otimes \cdots \otimes u_N |\psi\rangle$

$\Rightarrow$ compatibility constraints

$\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq ... \geq \lambda_d^{(i)}$

depend only on eigenvalues $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_d^{(i)}) \in \mathbb{R}^{d-1}$

Goal: characterise compatible $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)}) \in \mathbb{R}^{N(d-1)}$

Warm-Up $N = 2$

singular values

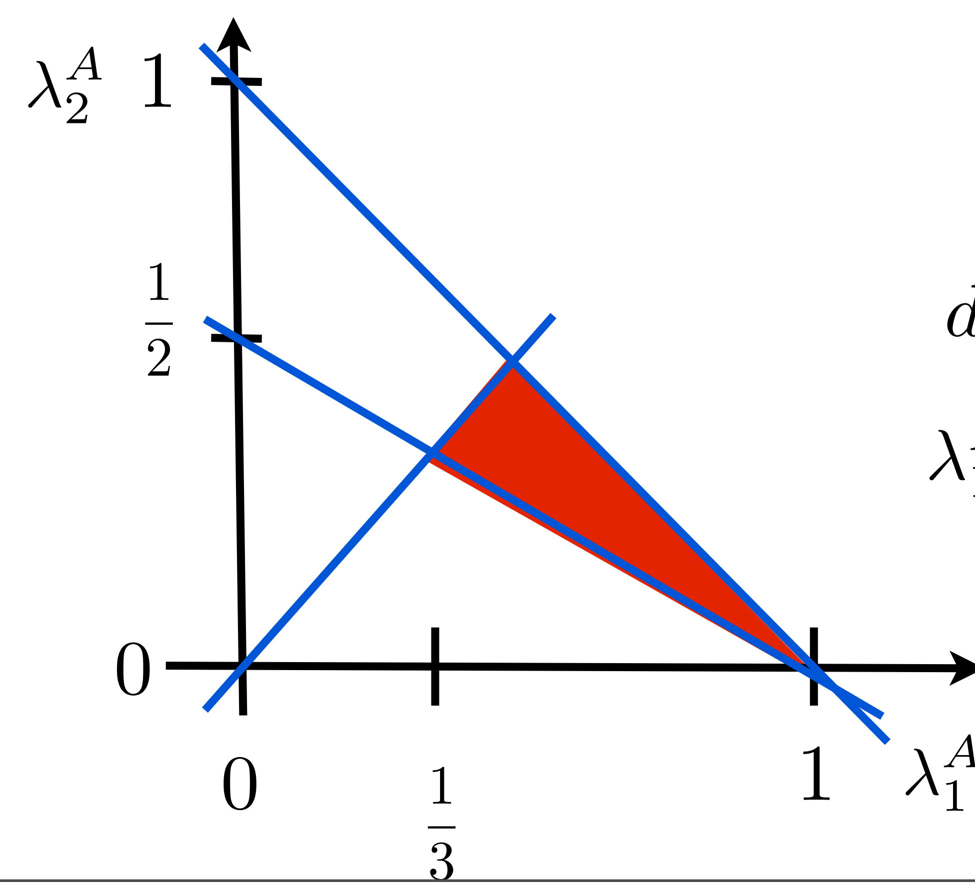
$$|\psi\rangle_{AB} = \sum_j s_j |e_j\rangle_A |f_j\rangle_B$$

$d = 2$

$\lambda_1^A \geq \lambda_2^A$

$$\Rightarrow \qquad \lambda_j^A = s_j^2 = \lambda_j^B$$

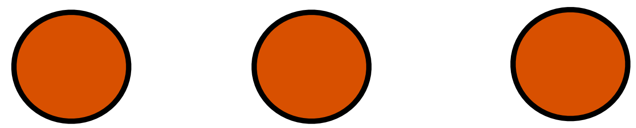
polytopes!



$d = 3$

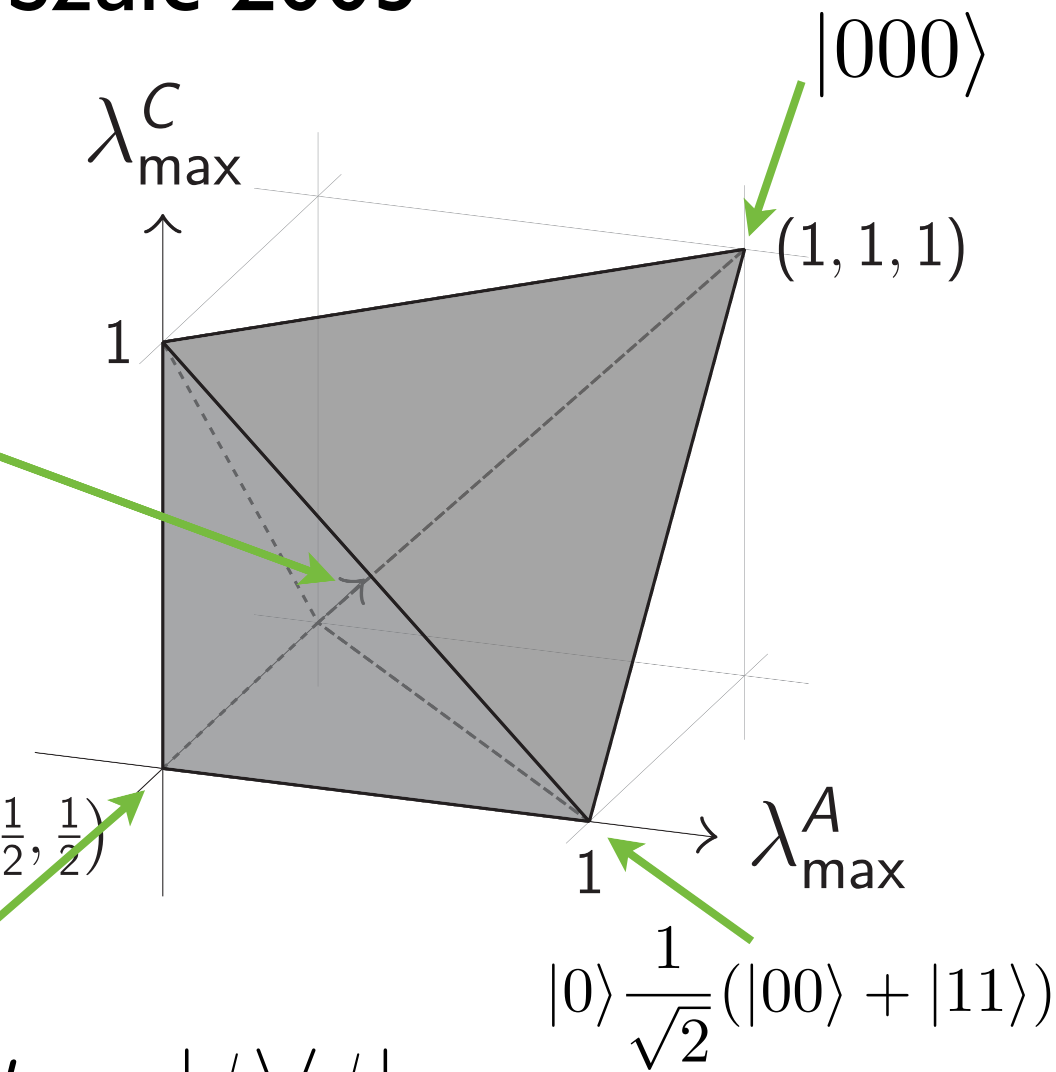
$\lambda_1^A \geq \lambda_2^A \geq \lambda_3^A$

$N = 3\ d = 2$ Higuchi, Sudbery& Szulc 2003



$$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$\lambda_1^A + \lambda_1^B \leq 1 + \lambda_1^C$$ and cyclic



$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$\begin{aligned}
\lambda_1^A + \lambda_1^B &= max_{\phi_A,\phi_B} tr\rho_A|\phi\rangle\langle\phi|_A + tr\rho_B|\phi\rangle\langle\phi|_B \\
&= max_{\phi_A,\phi_B} tr\rho_{AB}(|\phi\rangle\langle\phi|_A \otimes id + id \otimes |\phi\rangle\langle\phi|_B) \\
&\leq max_{\phi_A,\phi_B} tr\rho_{AB}(id \otimes id + |\phi\rangle\langle\phi|_A \otimes |\phi\rangle\langle\phi|_B) \\
&\leq 1 + max_{\phi_{AB}} tr\rho_{AB}|\phi\rangle\langle\phi|_{AB} \\
&= 1 + \lambda_1^{AB} = 1 + \lambda_1^C
\end{aligned}$$

inclusion/ exclusion

polytope! geometry related to entanglement!?

Sufficiency

Ansatz $$|\psi\rangle_{ABC} = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle$$

$$\rho_A = |0\rangle\langle 0| \quad \rho_B = |0\rangle\langle 0| \quad \rho_C = |0\rangle\langle 0|$$

$$\rho_A = |0\rangle\langle 0| \quad \rho_B = |1\rangle\langle 1| \quad \rho_C = |1\rangle\langle 1|$$

$$\rho_A = |1\rangle\langle 1| \quad \rho_B = |0\rangle\langle 0| \quad \rho_C = |1\rangle\langle 1|$$

$$\rho_A = |1\rangle\langle 1| \quad \rho_B = |1\rangle\langle 1| \quad \rho_C = |0\rangle\langle 0|$$

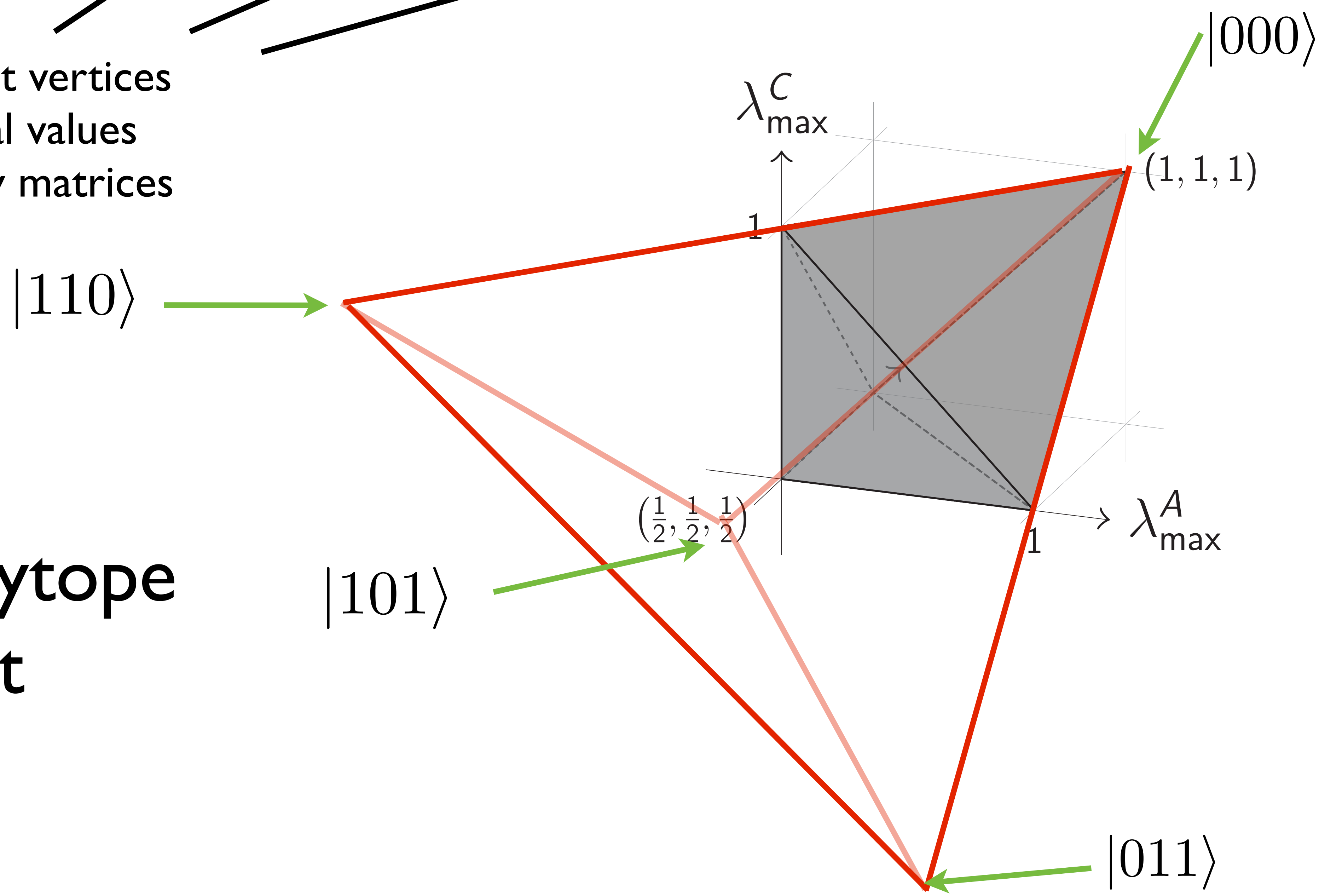
since pairs differ in two positions each, local density matrices given by convex combinations

$$\rho_A = (a^2+b^2)|0\rangle\langle 0| + (c^2+d^2)|1\rangle\langle 1|$$
$$\rho_B = (a^2+c^2)|0\rangle\langle 0| + (b^2+d^2)|1\rangle\langle 1|$$
$$\rho_C = (a^2+d^2)|0\rangle\langle 0| + (b^2+c^2)|1\rangle\langle 1|$$

Graphically

$$|\psi\rangle_{ABC} = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle$$

map to outer most vertices
of set of diagonal values
of reduced density matrices



eigenvalue polytope
=first quadrant