Exercise 1. ***Pusey–Barrett–Rudolph argument for non-orthogonal states***

In this exercise you will generalize the argument given in the lecture to arbitrary non-orthogonal states by following the appendix of the paper by Pusey, Barrett and Rudolph (arXiv:1111.3328). For this, suppose that you are given n copies of a device that prepares a quantum system in either the state $|\psi_0\rangle$ or the state $|\psi_1\rangle$. Thus there are 2^n possible preparations, $|\psi_{\vec{x}}\rangle = |\psi_{x_1}\rangle \otimes \ldots \otimes |\psi_{x_n}\rangle$, where $\vec{x}$ is a binary string with $x_i = 0/1$ if the i-th system is prepared in state $|\psi_{0/1}\rangle$. The challenge is to find a joint measurement of the n systems such that each measurement outcome has probability zero for (at least) one of the preparations.

(a) Show that (up to global phases) there always exist orthonormal vectors $|0\rangle$, $|1\rangle$ such that

$$|\psi_0\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,|1\rangle, \qquad |\psi_1\rangle = \cos\frac{\theta}{2}\,|0\rangle - \sin\frac{\theta}{2}\,|1\rangle,$$

with $0 < \theta \leq \frac{\pi}{2}$.

We shall think of $|0\rangle$ and $|1\rangle$ defining the computational basis of a qubit, and write $|\vec{x}\rangle = |x_1\rangle \otimes \ldots \otimes |x_n\rangle$ for the corresponding product basis.

(b) Consider the following measurement procedure: First, apply the unitary $Z_\beta = \left(\begin{smallmatrix} 1 & \\ & e^{i\beta} \end{smallmatrix}\right)$ to each qubit. Second, apply the unitary R_α which maps $|0\ldots 0\rangle \mapsto e^{i\alpha}|0\ldots 0\rangle$ and acts as the identity on the orthogonal complement. Third, apply a Hadamard gate $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ to each qubit. Finally, measure each qubit in the computational basis. Show that the probability of outcome $|\vec{x}\rangle$ given the initial state $|\psi_{\vec{x}}\rangle$ as predicted by quantum mechanics is equal to

$$\frac{1}{2^n}\left(\cos\frac{\theta}{2}\right)^{2n}\left|e^{i\alpha} - 1 + \left(1 + e^{i\beta}\tan\frac{\theta}{2}\right)^n\right|^2.$$

(c) Show that for large enough n, the phases α, β can be chosen such that these probabilities are all zero.

(d) Conclude that, under the assumptions 1–3 stated in the lecture, there does not exist a physical state z compatible with both $|\psi_0\rangle$ and $|\psi_1\rangle$.

Exercise 2. ***Reality of the wave function from different assumptions***

In this exercise, your task is to understand an alternative argument due to Colbeck and Renner (arXiv:1111.6597), which derives the reality of the wave function from a different set of assumptions. As in the lecture, we consider a system prepared in a state described by a wave function Ψ; the experimenter then chooses a measurement setting A and records the measurement outcome X. Mathematically, Ψ, A, and X are modelled by random variables on some underlying probability space. Let $\Gamma \ni \Psi$ be a collection of random variables on the same probability space, modeling all information that is in principle available before the measurement setting is chosen. Technically, we shall assume that *measurement settings can be chosen freely*; in particular, this implies that $\mathbb{P}_{A|\Gamma} = \mathbb{P}_A$. We shall also assume that *quantum theory is correct*, so that e.g. $\mathbb{P}_{X|A\Psi}$ is given by the predictions of quantum mechanics.

Under these assumptions, it has been shown that the wave function Ψ is complete for the description of this system (arXiv:1005.5173). Here and in the following, a subset of random variables $\Gamma_0 \subseteq \Gamma$ is said to be *complete* for the description of the system if $\mathbb{P}_{X|\Gamma A} = \mathbb{P}_{X|\Gamma_0 A}$, i.e. if

$$\Gamma \to (\Gamma_0, A) \to X$$

is a Markov chain.

(a) Compare this notion of completeness with the one discussed in last semester's quantum information theory lecture. *(This part of the exercise is optional.)*

Let us now consider another subset of random variables $Z \subseteq \Gamma$ (a "list of elements of reality") that is also complete for the description of the system.

(b) Show that

$$\mathbb{P}_{X|Z=z,A=a} = \mathbb{P}_{X|\Psi=\psi,A=a}$$

whenever $\mathbb{P}(Z = z, \Psi = \psi) > 0$ and $\mathbb{P}(A = a) > 0$.

(c) Conclude that the wave function Ψ is determined uniquely by Z. In this sense, the system's wave function is in one-to-one correspondence with its elements of reality.