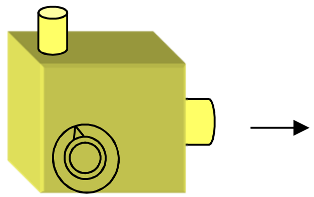
A generalized notion of noncontextuality for any operational theory

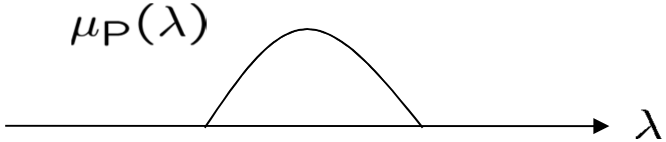
A hidden variable model of an operational theory

Specifies an ontic state space Λ

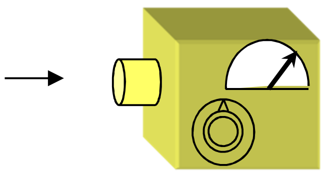
Preparation
P



$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$

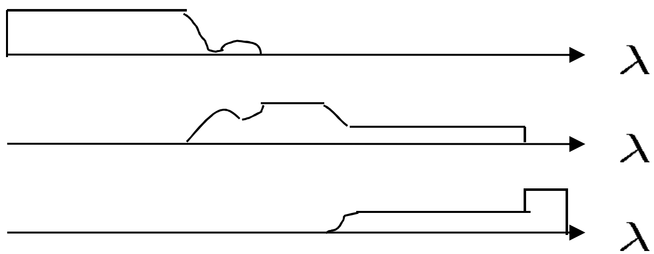


Measurement
M



$0 \leq \xi_{\mathsf{M},k} \leq 1$

$\sum_k \xi_{\mathsf{M},k}(\lambda) = 1$ for all λ



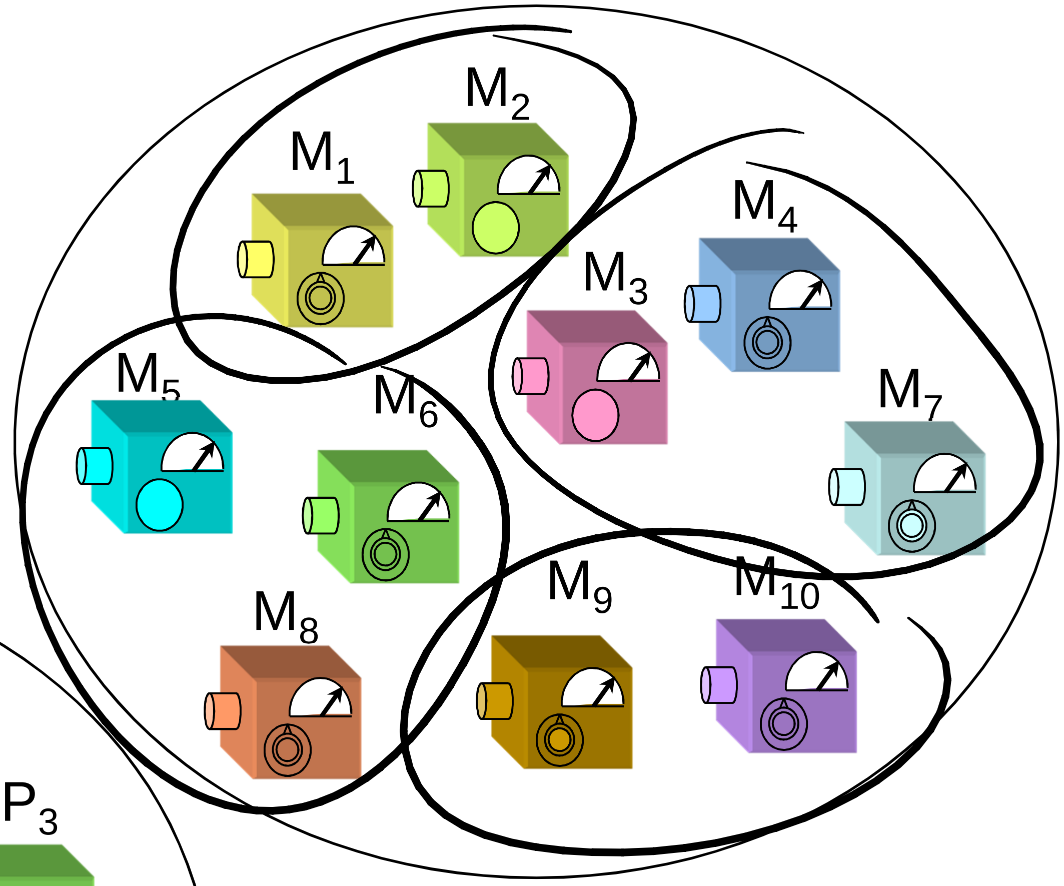
$$p(k|\mathsf{P},\mathsf{M}) = \int d\lambda \; \xi_{\mathsf{M},k}(\lambda) \; \mu_{\mathsf{P}}(\lambda)$$

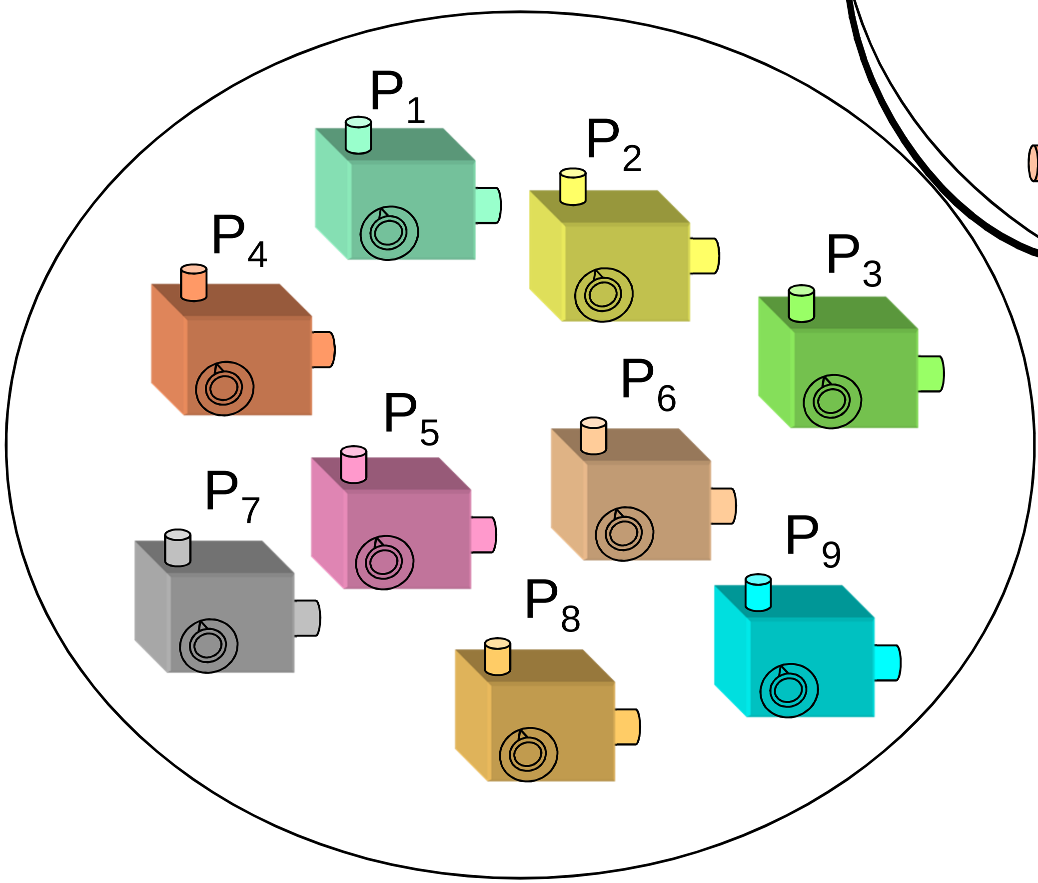
Generalized definition of noncontextuality:

A hidden variable model of an operational theory is noncontextual if

Operational equivalence of two experimental procedures → Equivalent representations in the hidden variable model

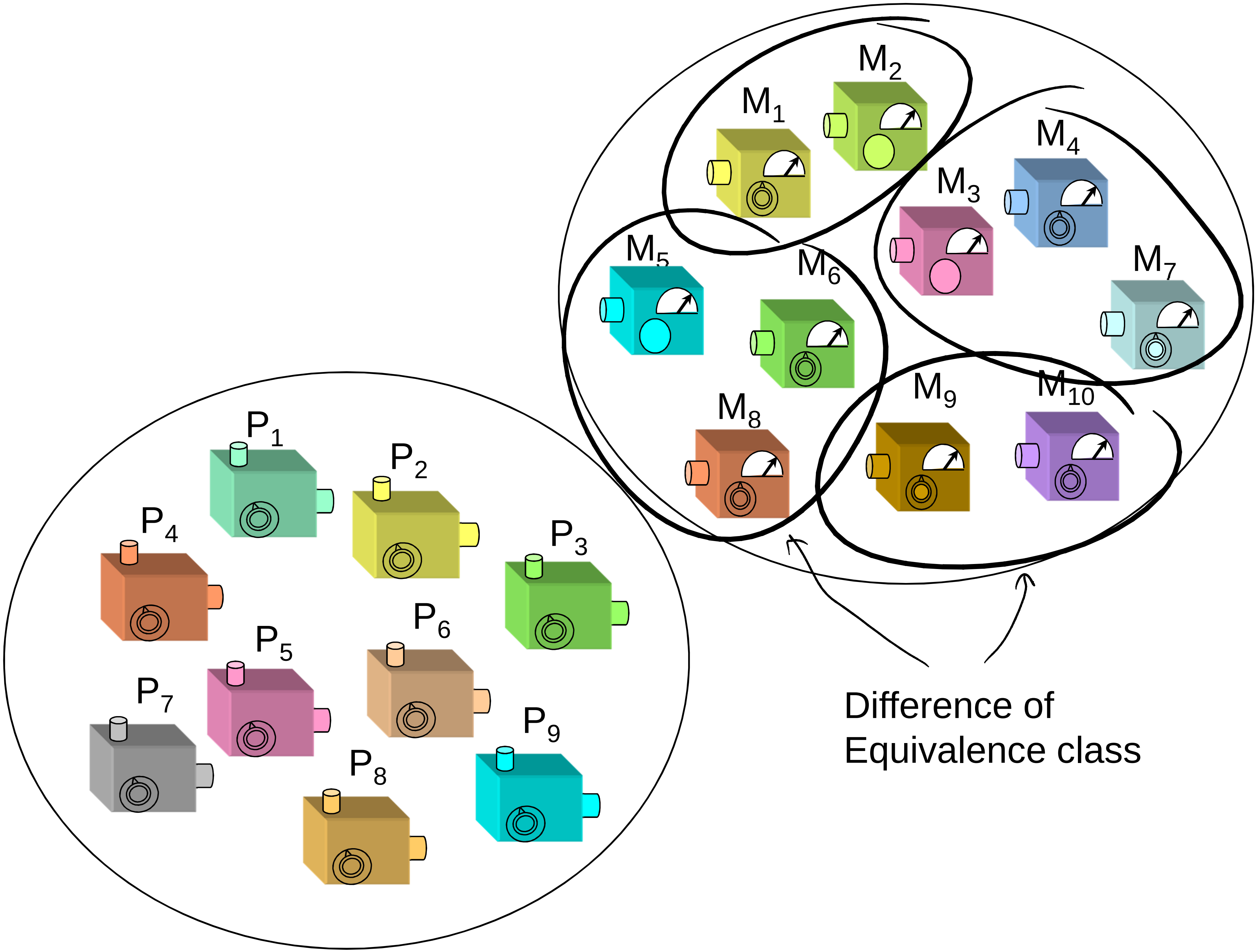
Measurement noncontextuality

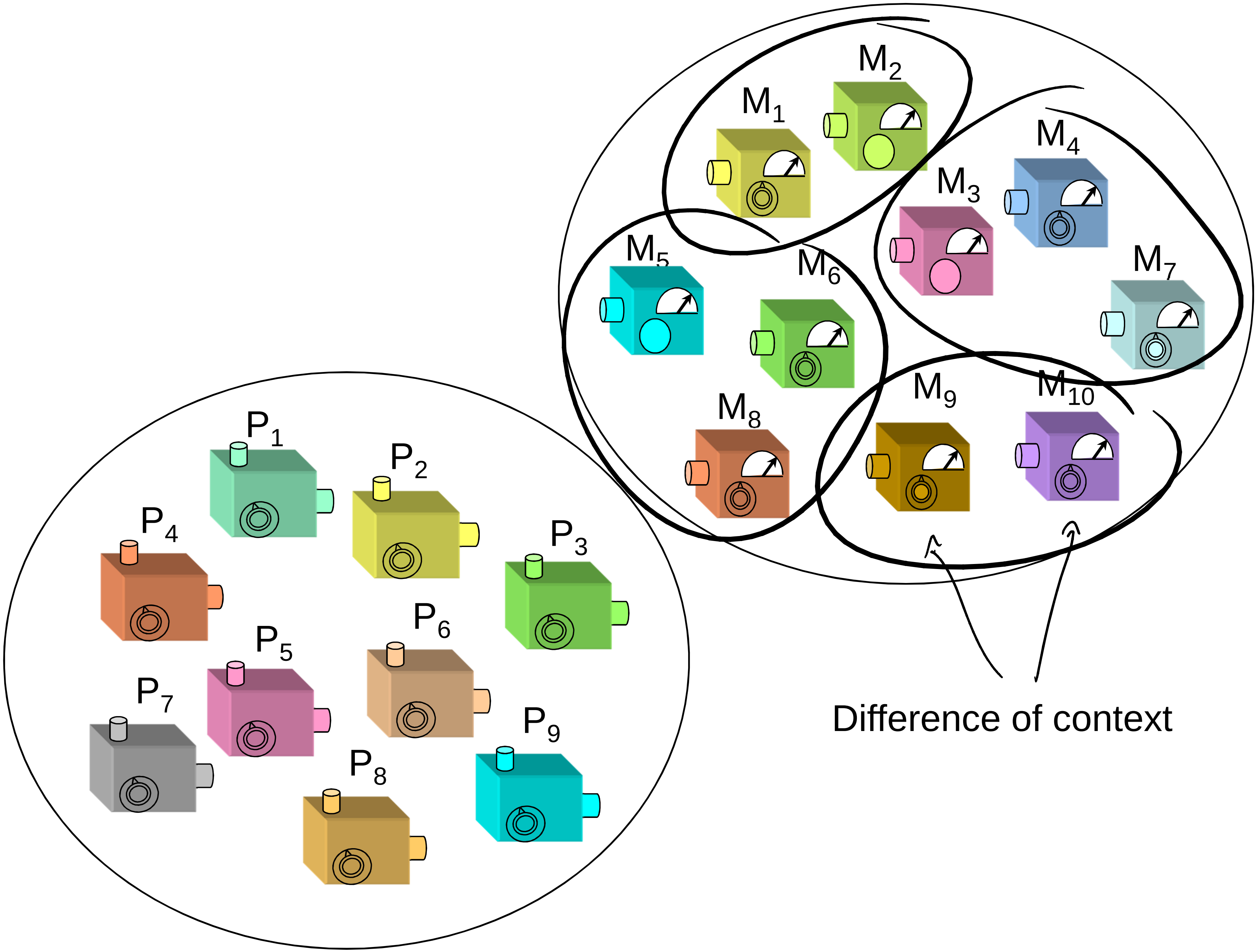


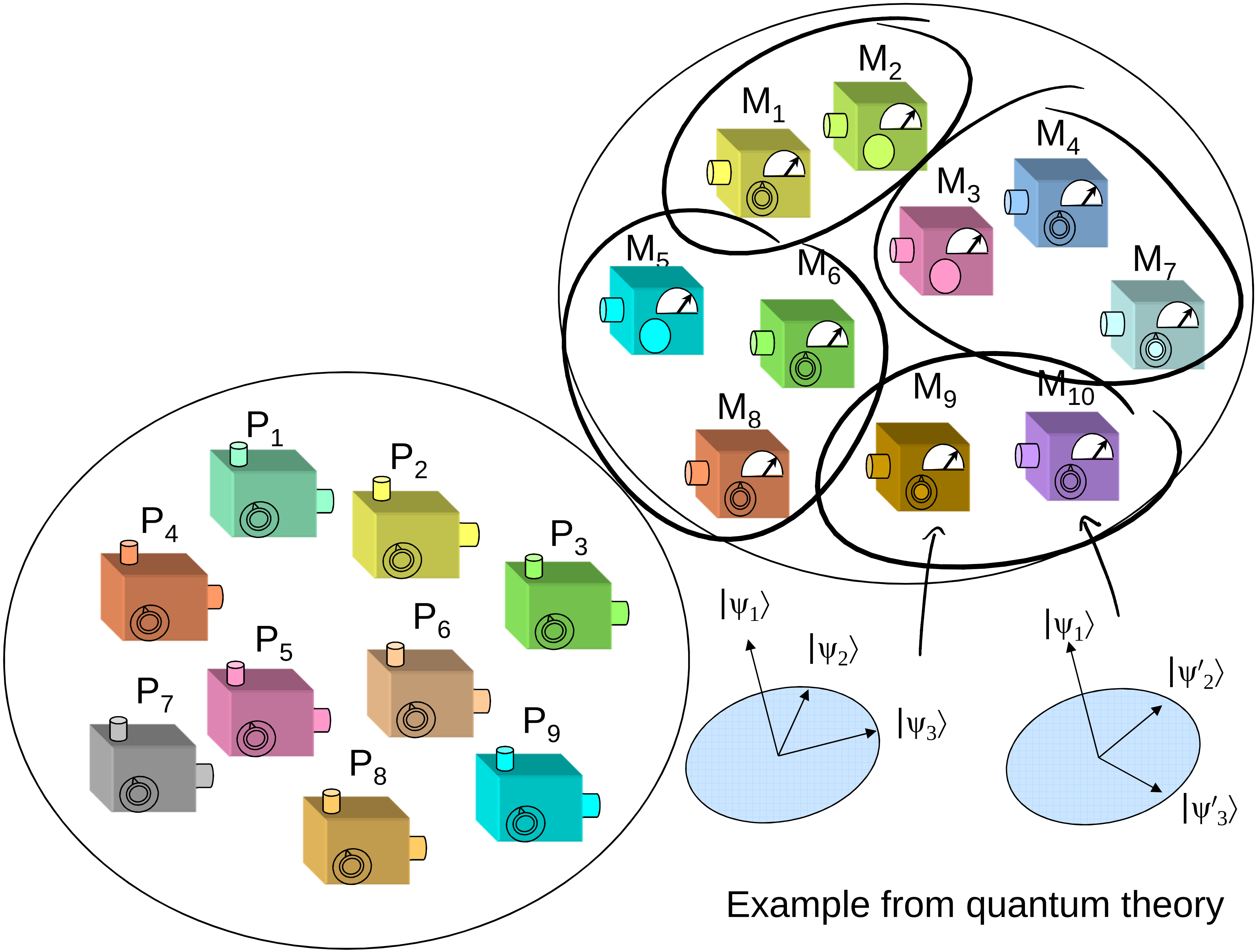


Operational equivalence classes

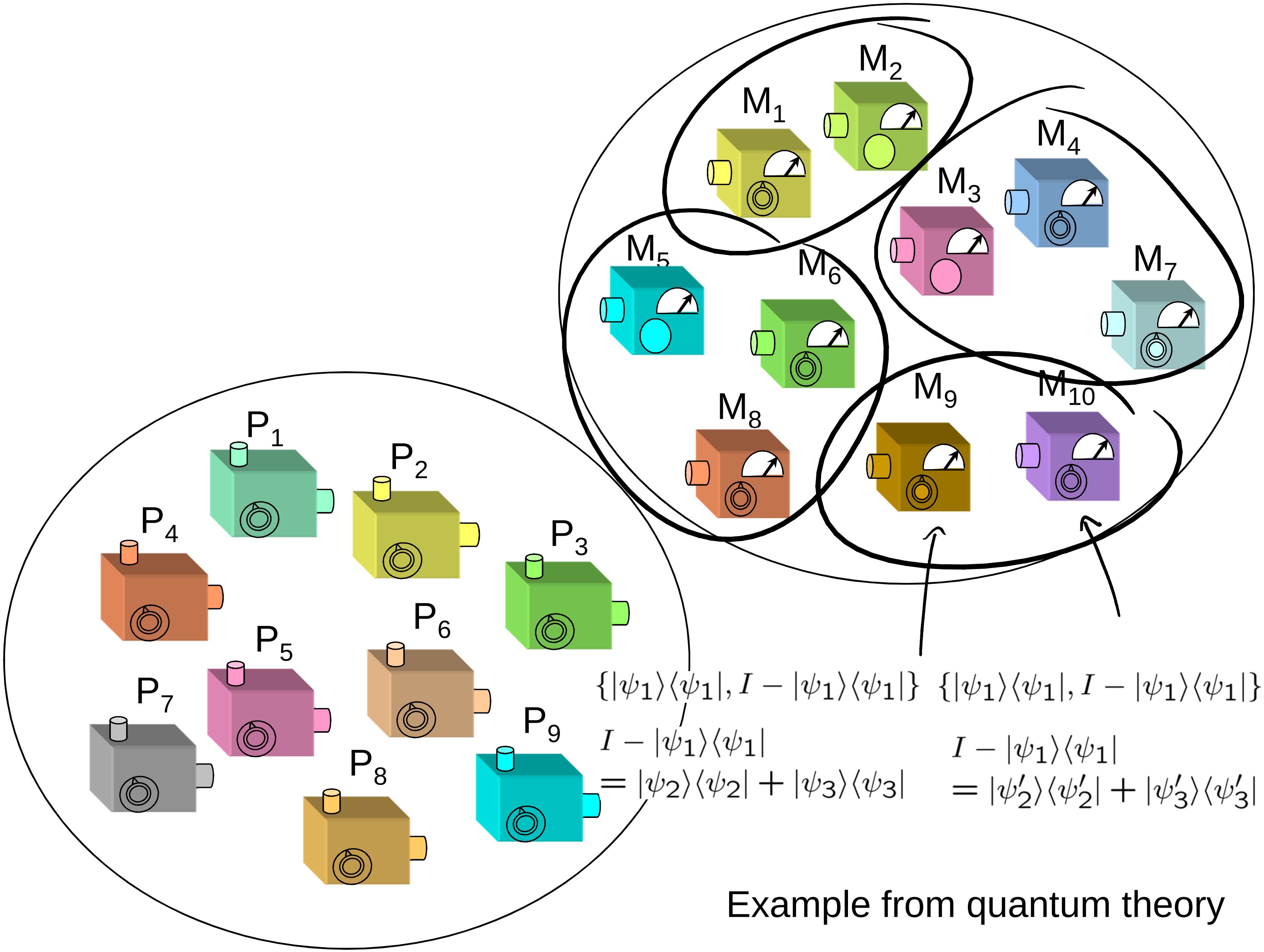
M is equivalent to M′ if
$\forall \mathsf{P} : p(k|\mathsf{P}, \mathsf{M}) = p(k|\mathsf{P}, \mathsf{M}')$


M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9
Difference of
Equivalence class

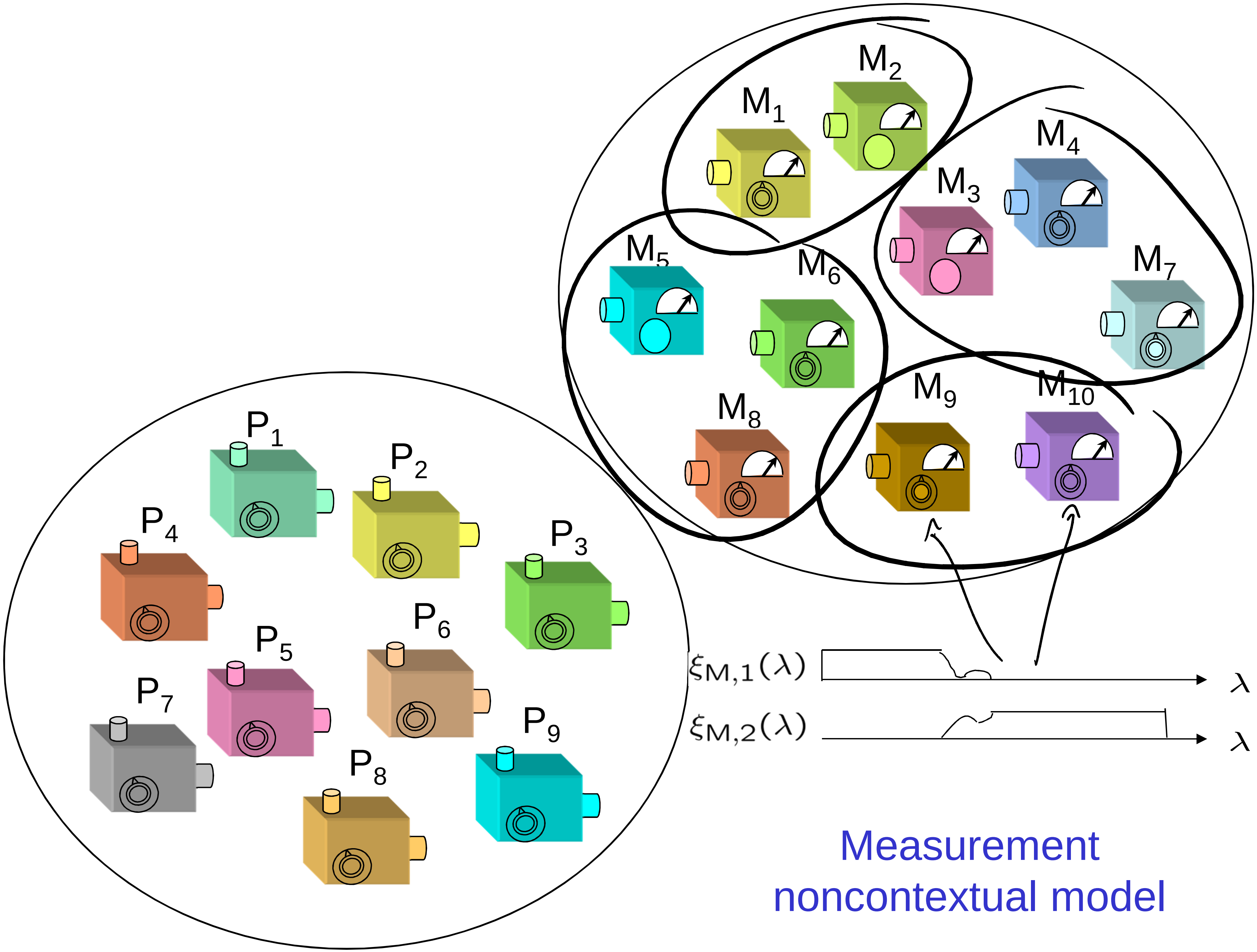

M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9
Difference of context



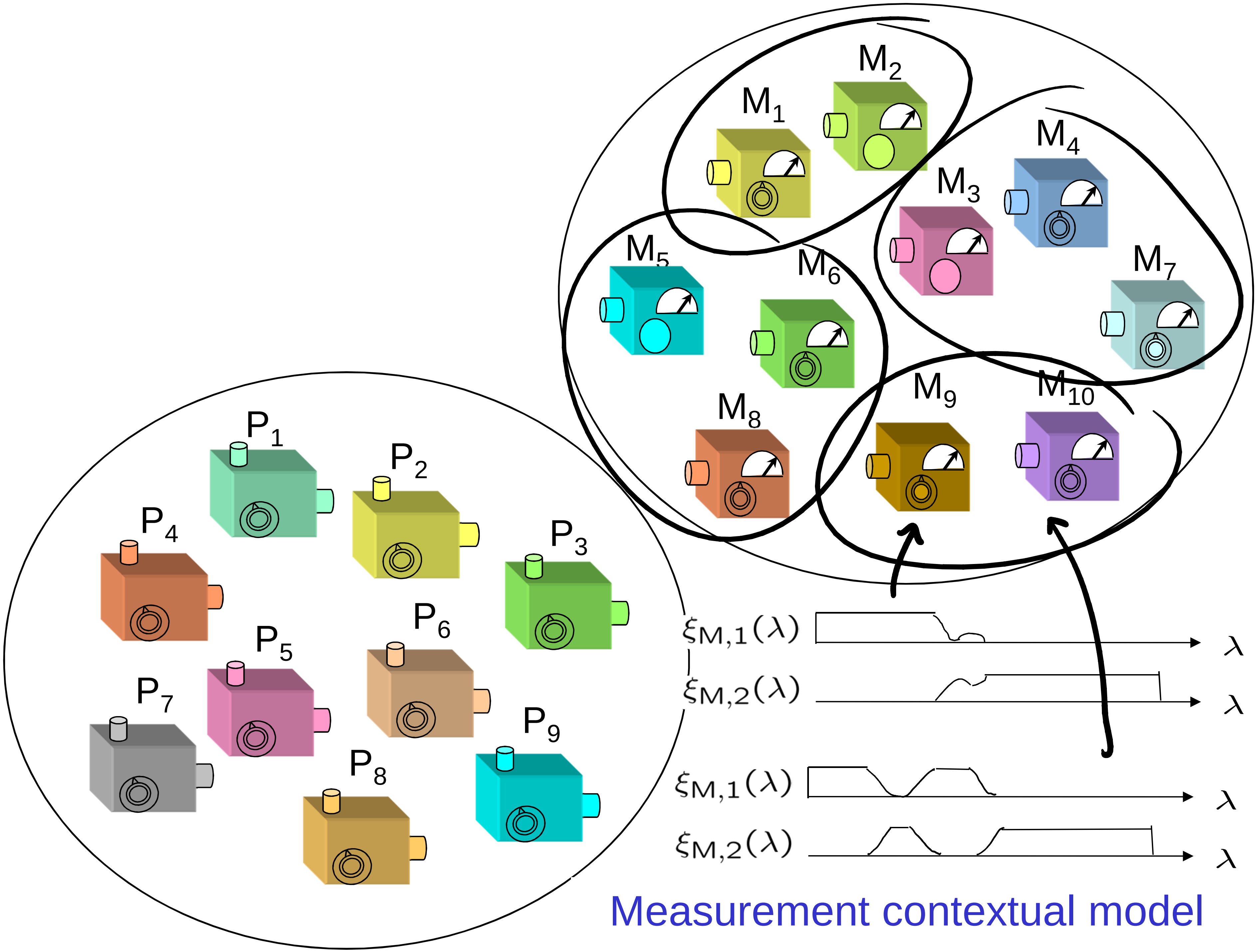
Example from quantum theory



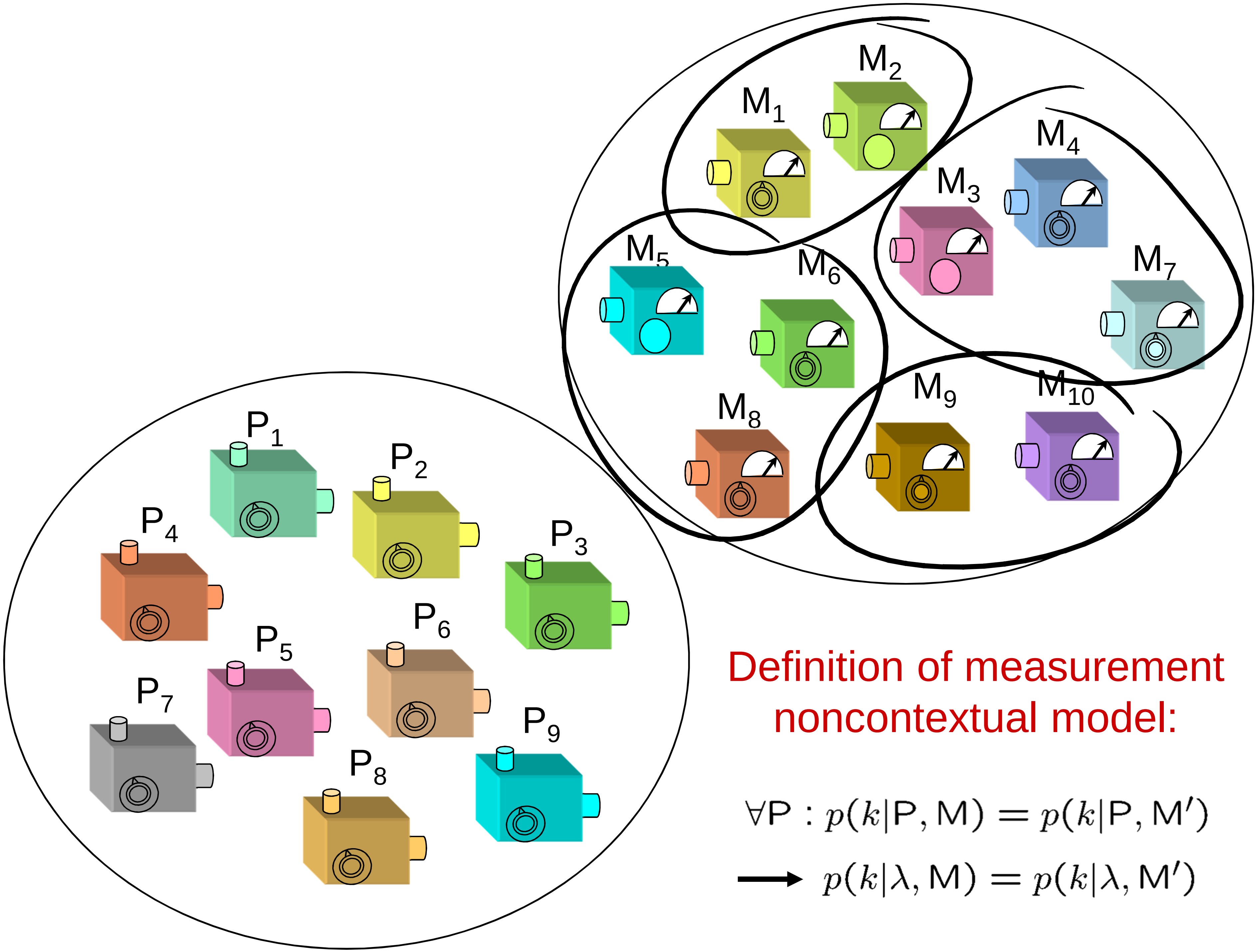
Example from quantum theory



Measurement
noncontextual model



Measurement contextual model



Definition of measurement noncontextual model:

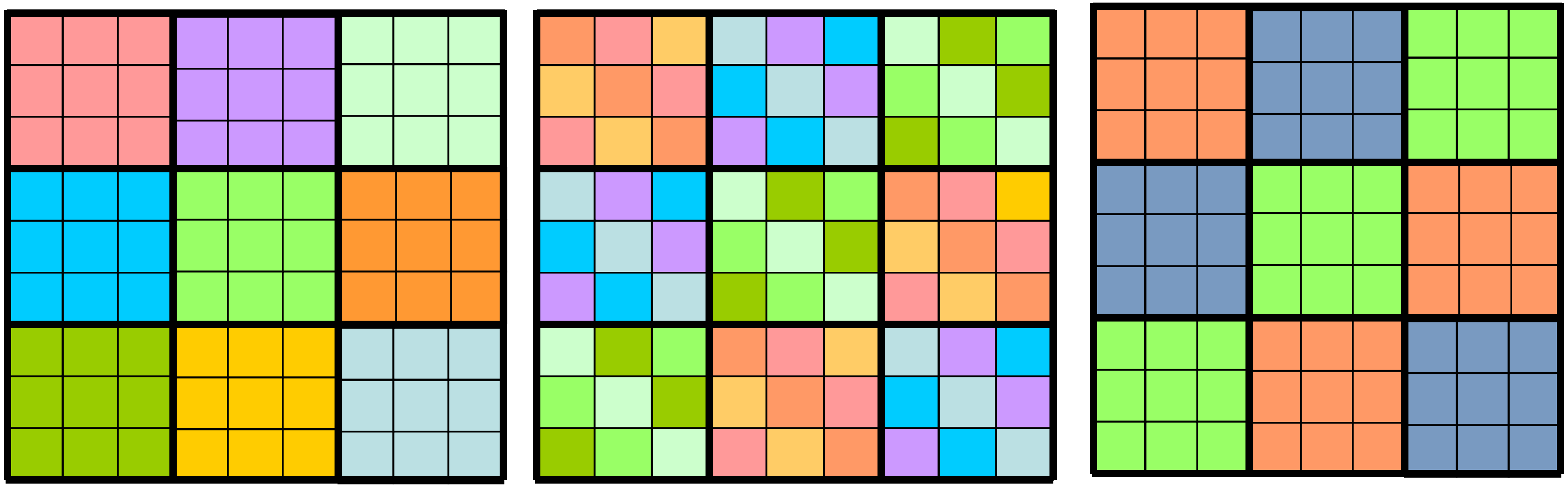
$$\forall \mathsf{P} : p(k|\mathsf{P}, \mathsf{M}) = p(k|\mathsf{P}, \mathsf{M}')$$

$$\longrightarrow p(k|\lambda, \mathsf{M}) = p(k|\lambda, \mathsf{M}')$$

Example of measurement noncontextual hidden variable model for a subtheory of quantum theory

$\|0\rangle\|0\rangle$	$\omega^0\|0\rangle\|0\rangle + \omega\|1\rangle\|1\rangle + \omega^*\|2\rangle\|2\rangle$	Π_0
$\|1\rangle\|1\rangle$	$\omega\|0\rangle\|0\rangle + \omega^*\|1\rangle\|1\rangle + \omega^0\|2\rangle\|2\rangle$	
$\|2\rangle\|2\rangle$	$\omega^*\|0\rangle\|0\rangle + \omega^0\|1\rangle\|1\rangle + \omega\|2\rangle\|2\rangle$	
$\|0\rangle\|1\rangle$	$\omega^0\|0\rangle\|1\rangle + \omega\|1\rangle\|2\rangle + \omega^*\|2\rangle\|0\rangle$	Π_1
$\|1\rangle\|2\rangle$	$\omega\|0\rangle\|1\rangle + \omega^*\|1\rangle\|2\rangle + \omega^0\|2\rangle\|0\rangle$	
$\|2\rangle\|0\rangle$	$\omega^*\|0\rangle\|1\rangle + \omega^0\|1\rangle\|2\rangle + \omega\|2\rangle\|0\rangle$	
$\|0\rangle\|2\rangle$	$\omega^0\|0\rangle\|2\rangle + \omega\|1\rangle\|0\rangle + \omega^*\|2\rangle\|1\rangle$	Π_2
$\|1\rangle\|0\rangle$	$\omega\|0\rangle\|2\rangle + \omega^*\|1\rangle\|0\rangle + \omega^0\|2\rangle\|1\rangle$	
$\|2\rangle\|1\rangle$	$\omega^*\|0\rangle\|2\rangle + \omega^0\|1\rangle\|0\rangle + \omega\|2\rangle\|1\rangle$	

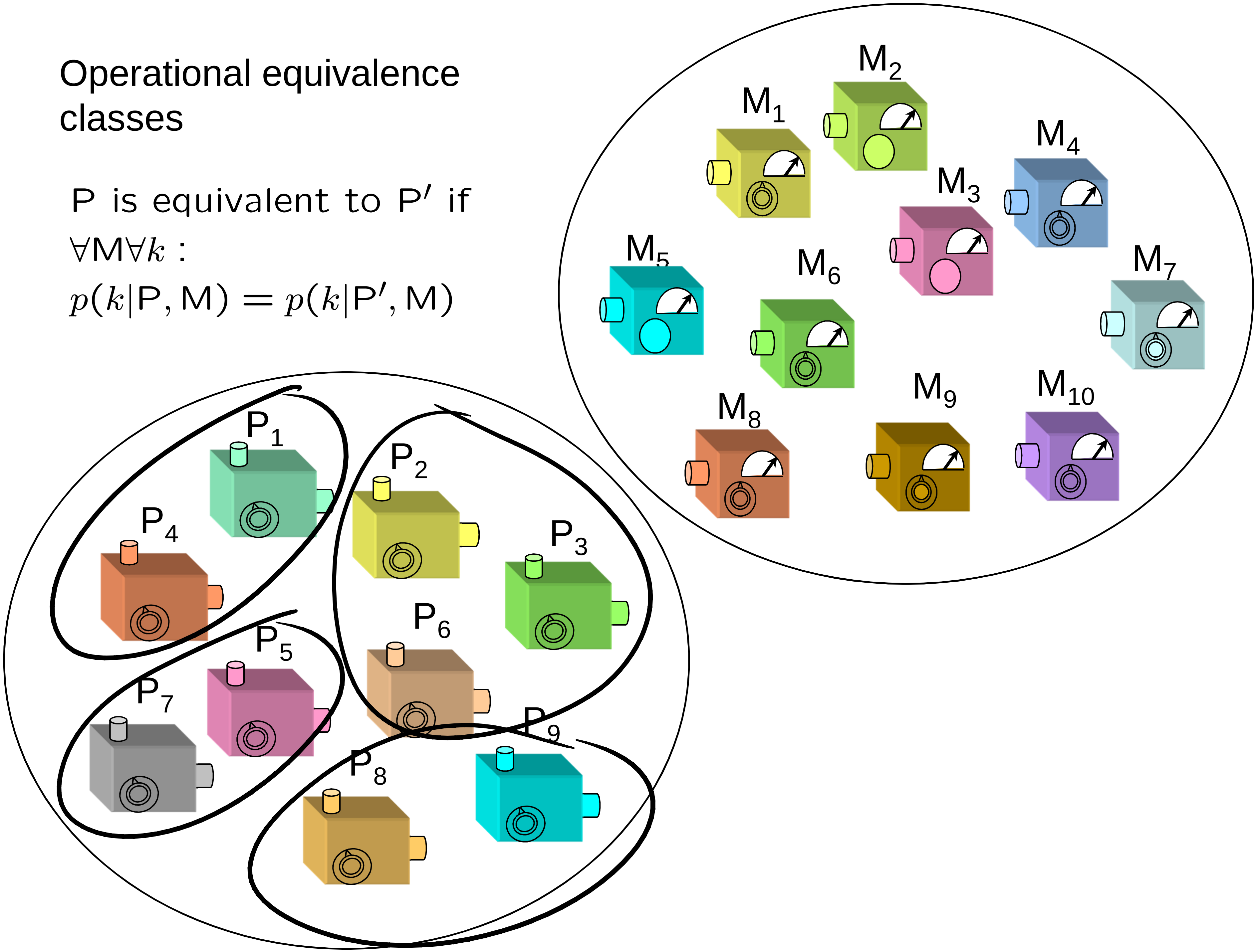
$$\omega = e^{i2\pi/3}$$

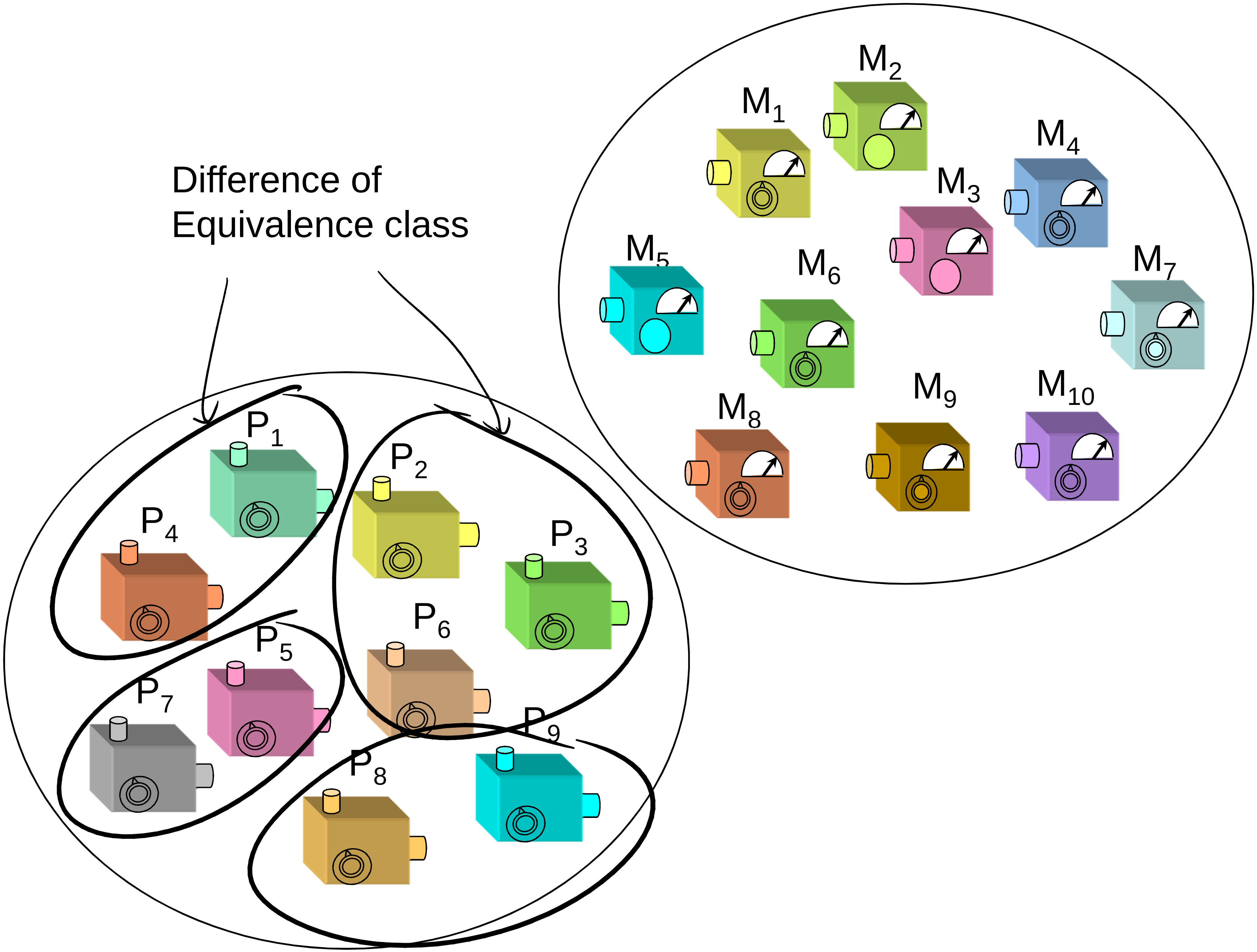


Preparation noncontextuality

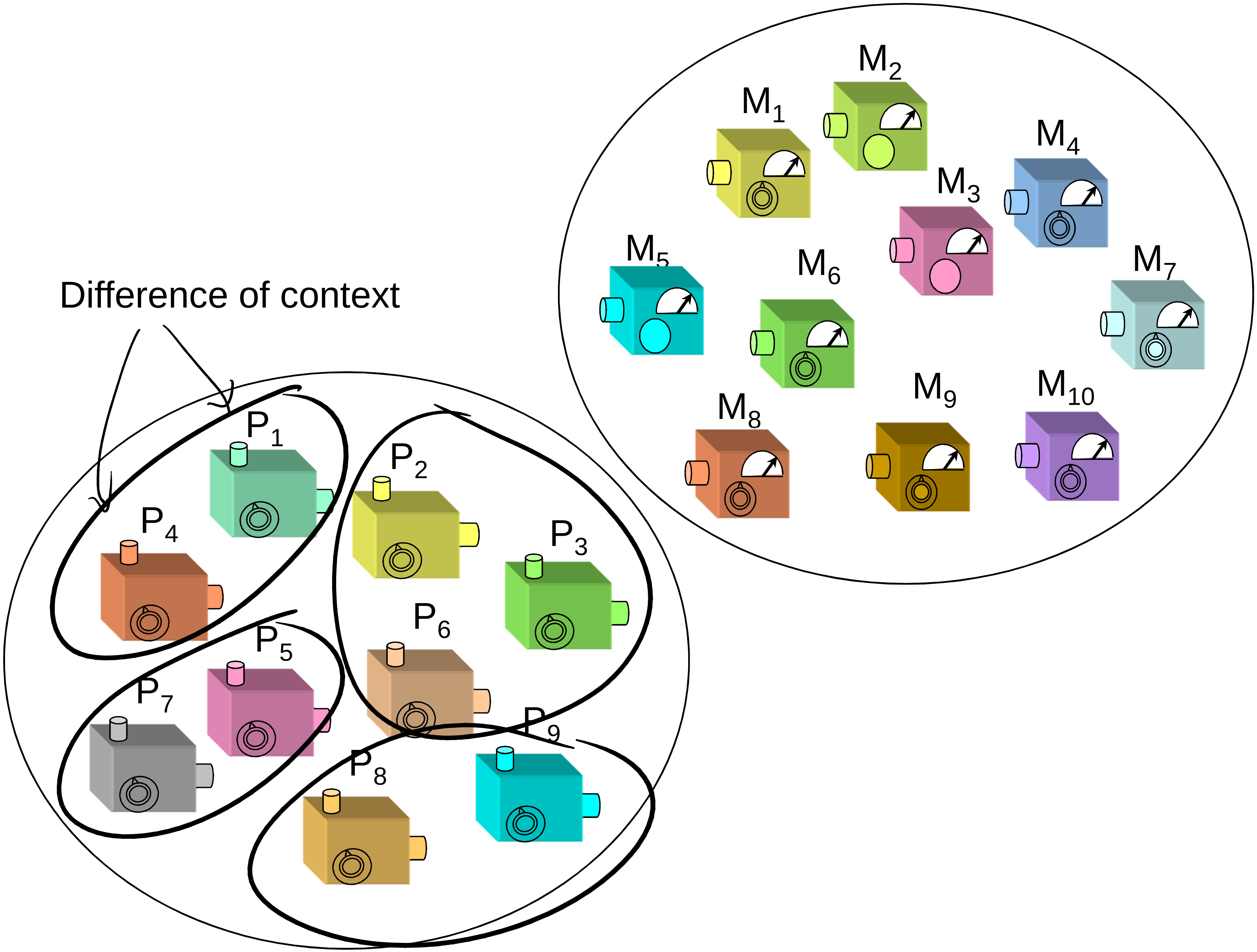
Operational equivalence classes

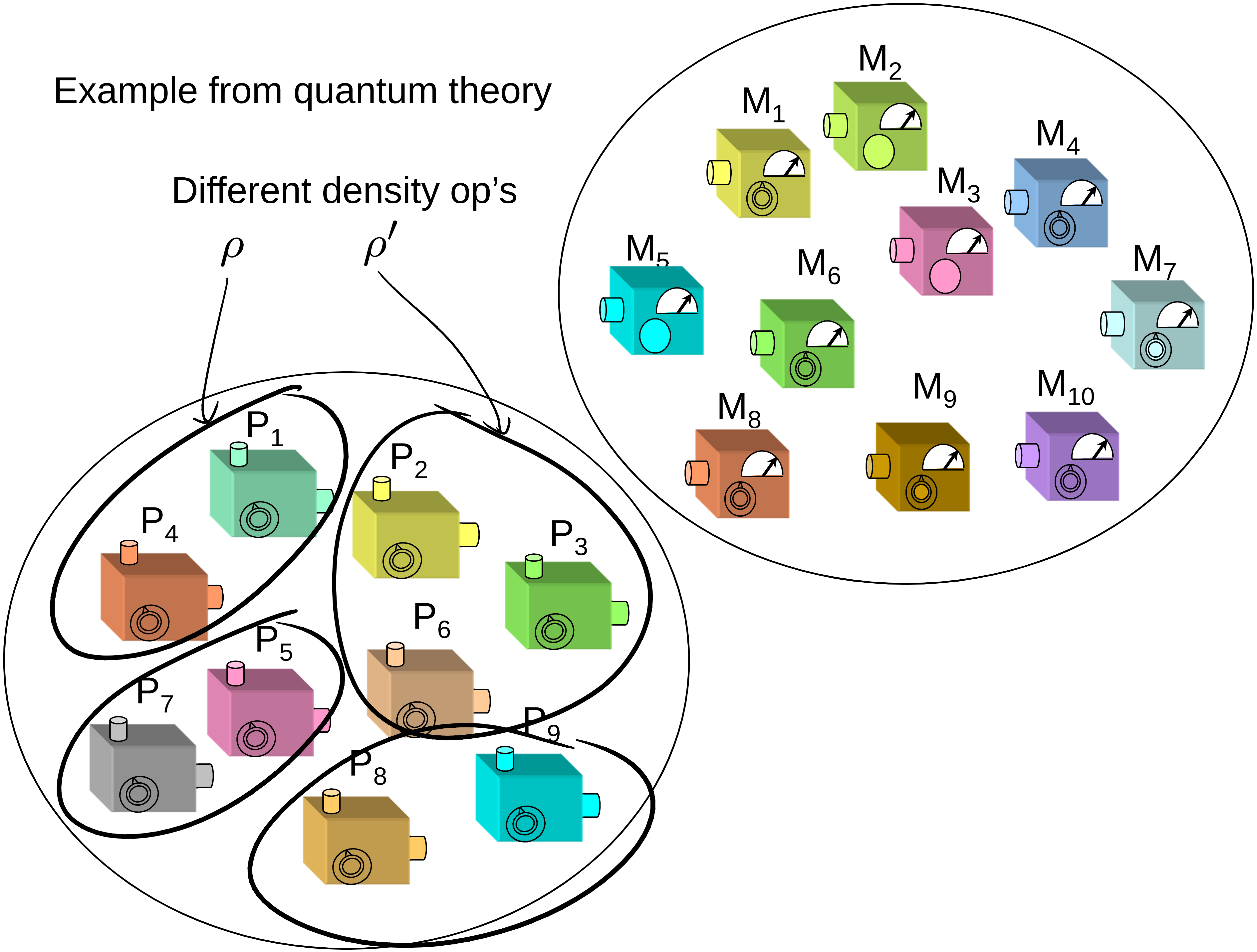
P is equivalent to P′ if
$\forall \mathsf{M} \forall k$:
$p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P}',\mathsf{M})$





Difference of
Equivalence class
M_1
M_2
M_3
M_4
M_5
M_6
M_7
M_8
M_9
M_{10}
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9

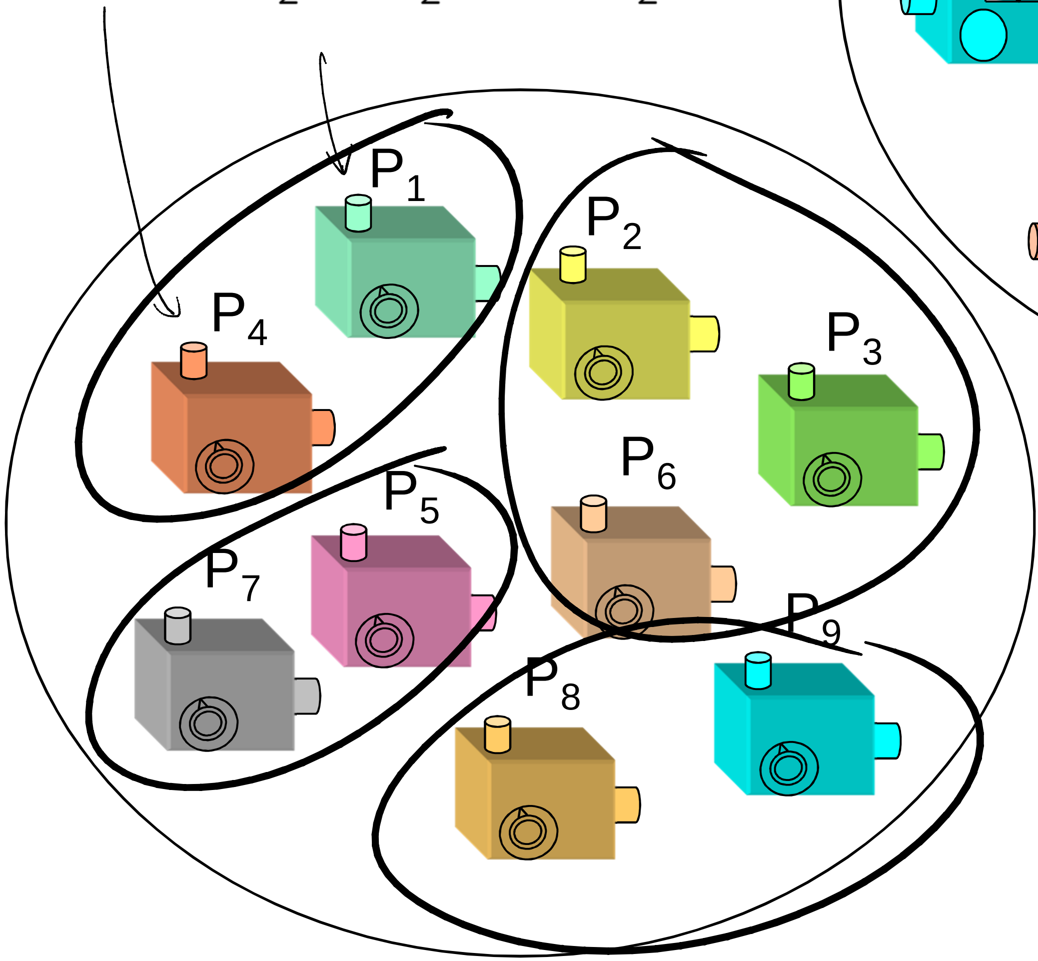

Difference of context
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
M_1
M_2
M_3
M_4
M_5
M_6
M_7
M_8
M_9
M_{10}

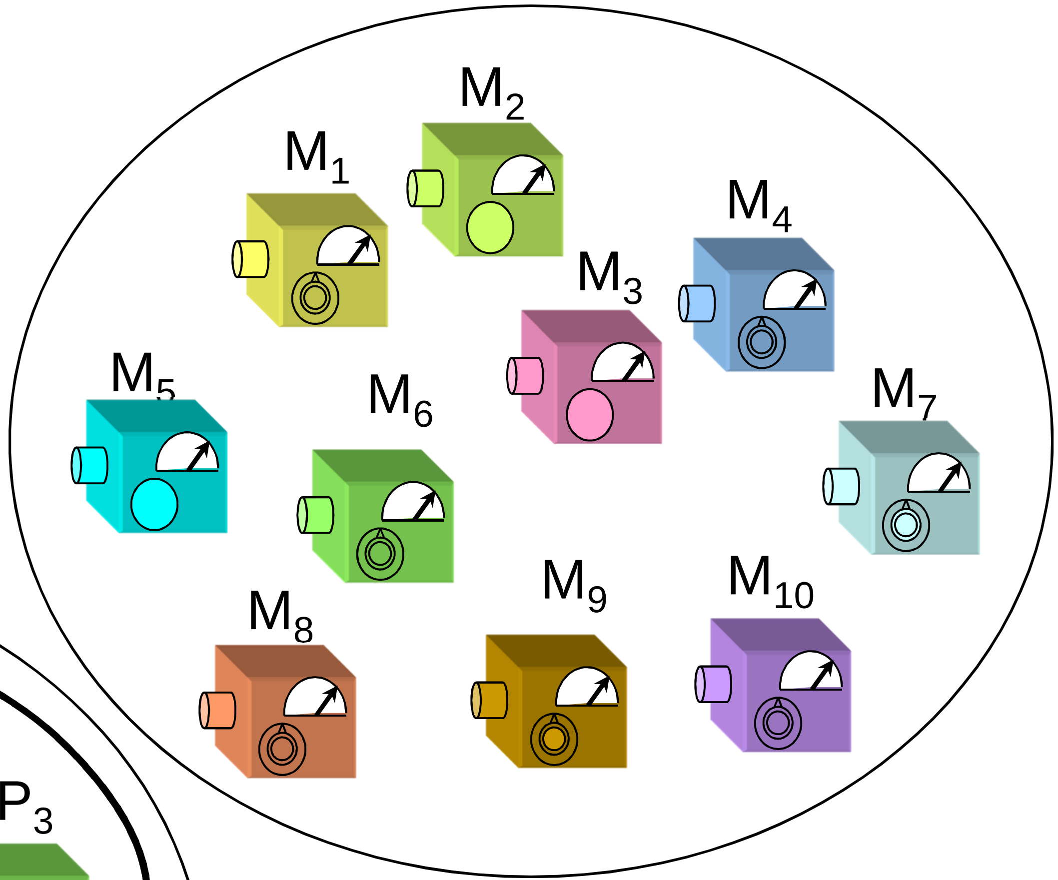

Example from quantum theory
Different density op's
ρ
ρ'
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
M_1
M_2
M_3
M_4
M_5
M_6
M_7
M_8
M_9
M_{10}

Example from quantum theory

$$\frac{1}{2}I = \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right|$$

$$\frac{1}{2}I = \frac{1}{2}\left|+\right\rangle\left\langle +\right| + \frac{1}{2}\left|-\right\rangle\left\langle -\right|$$

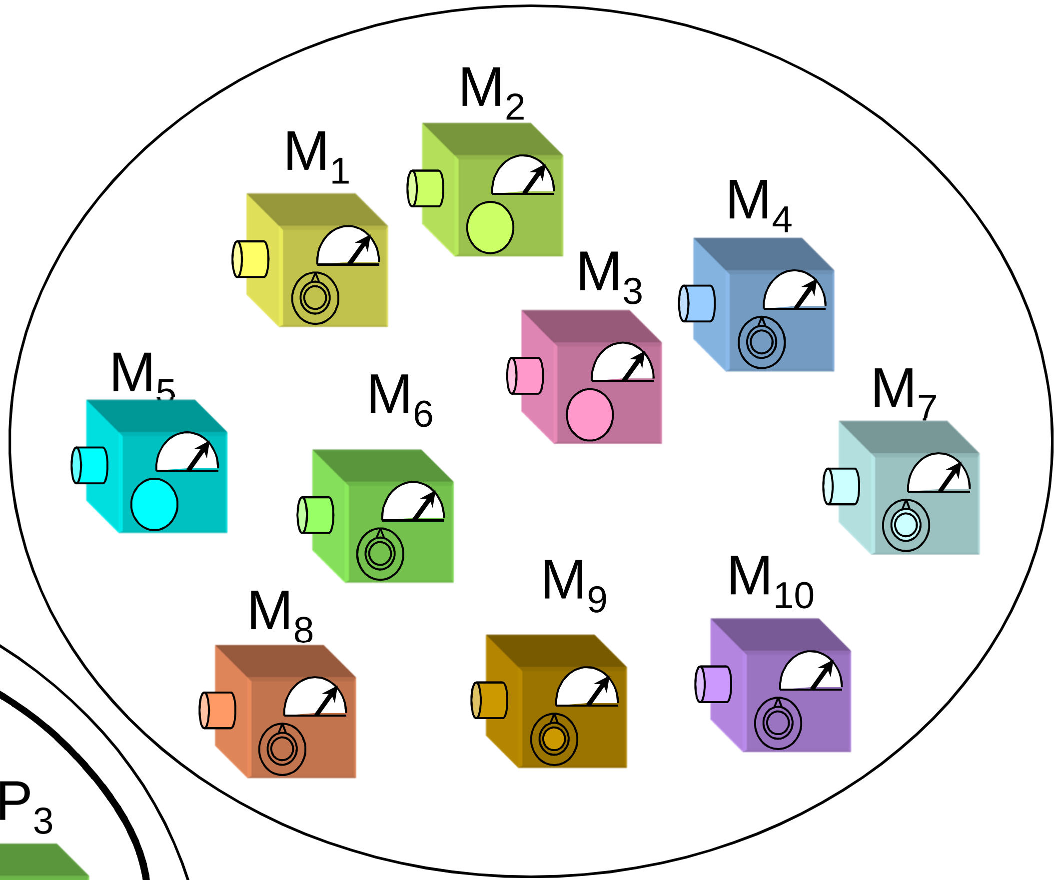


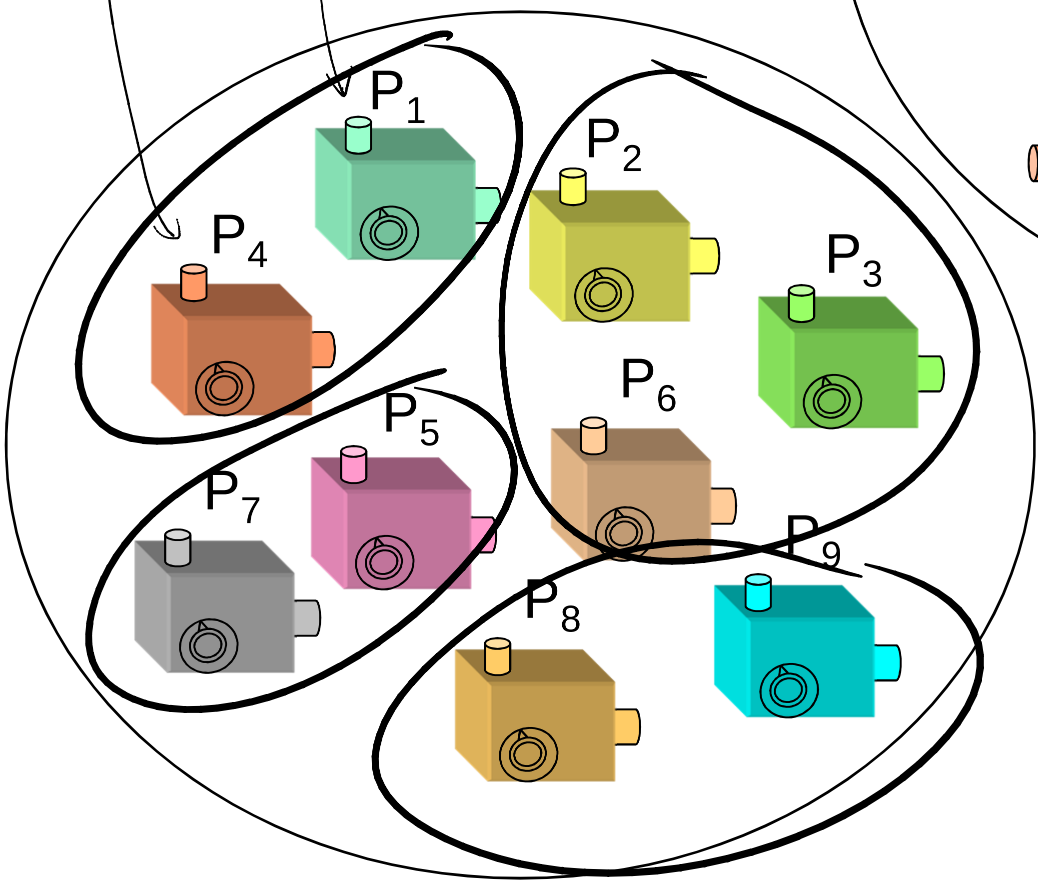


Example from quantum theory

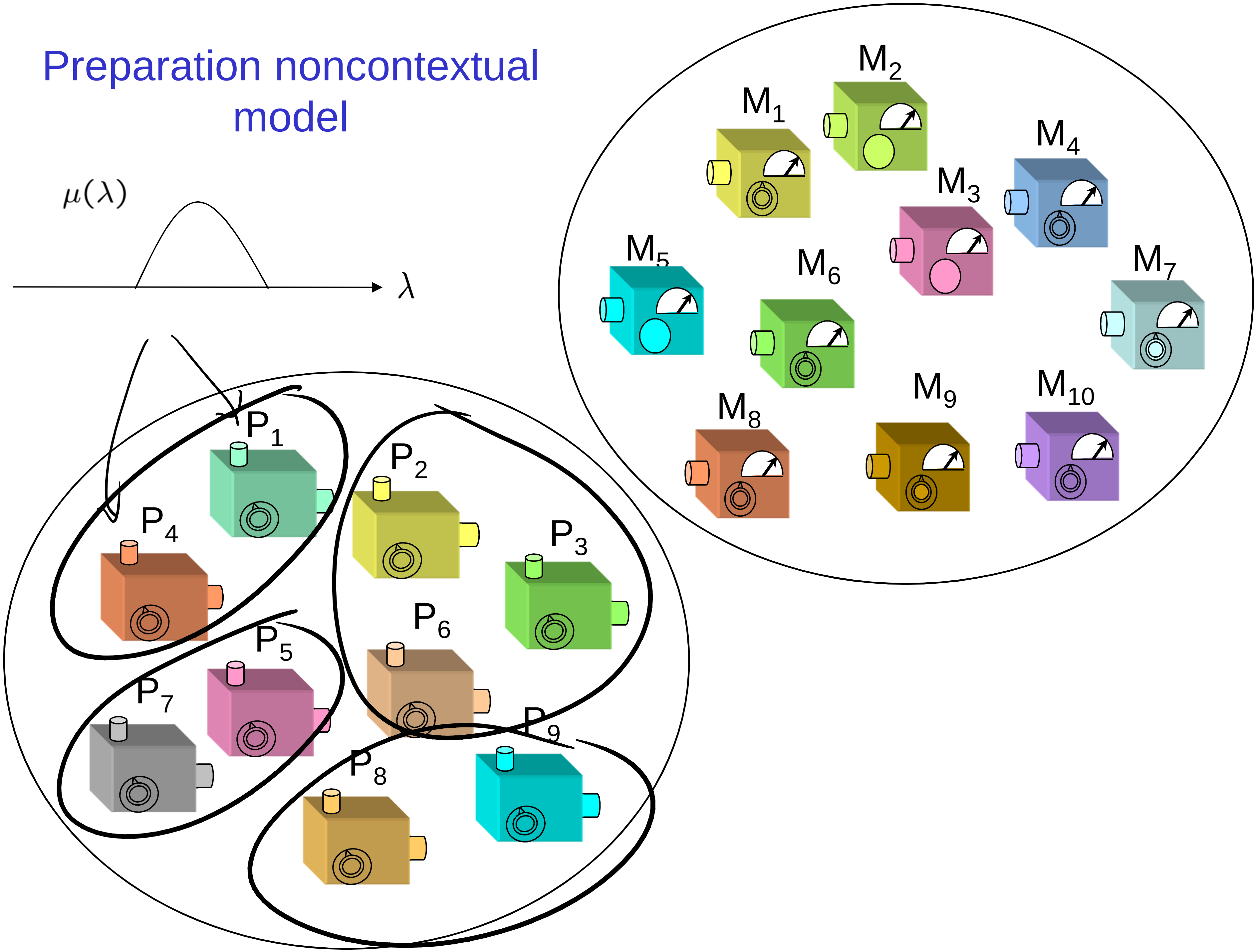
$$\frac{1}{2}I = \mathrm{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)]$$

$$\frac{1}{2}I = \mathrm{Tr}_B[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)]$$

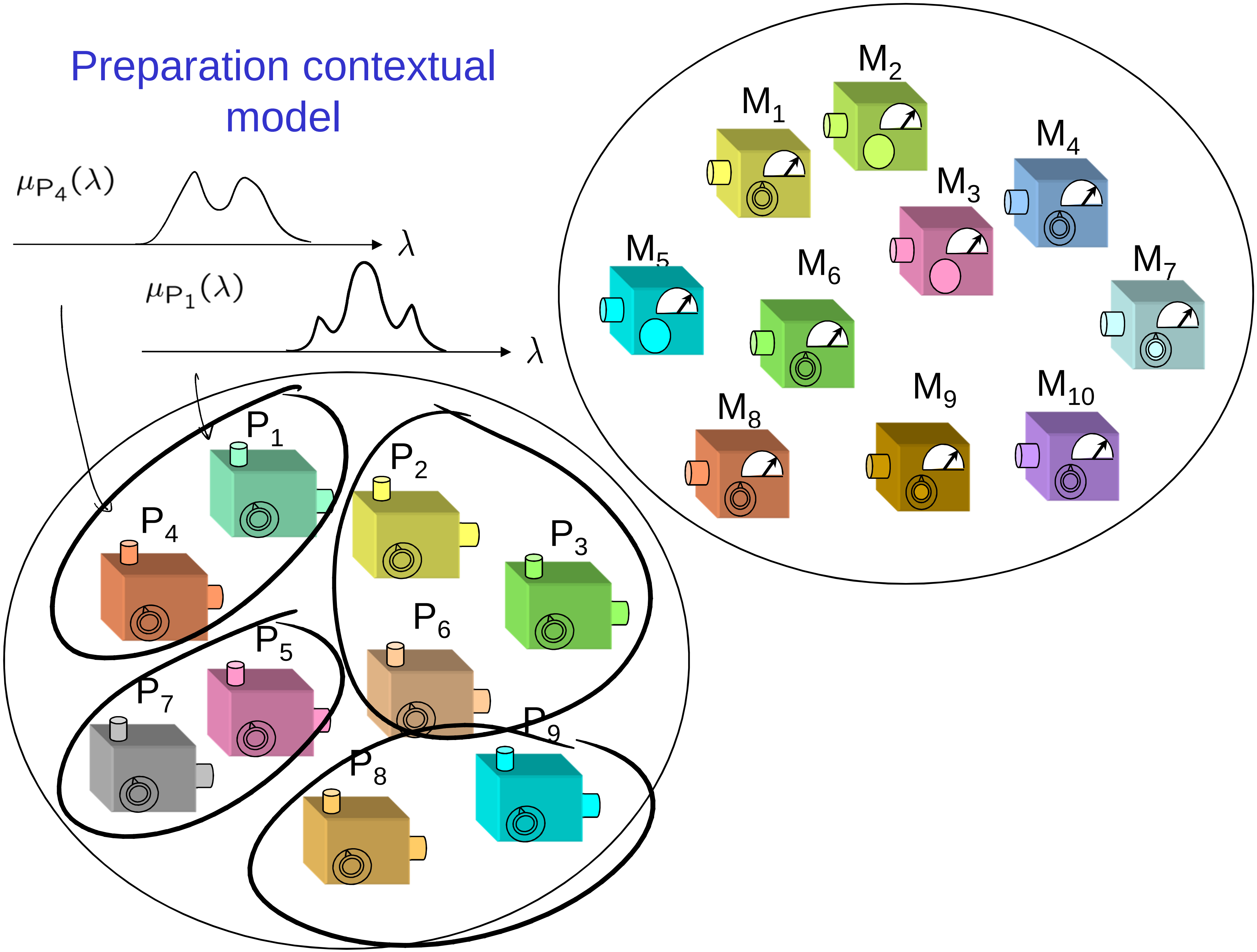




Preparation noncontextual model



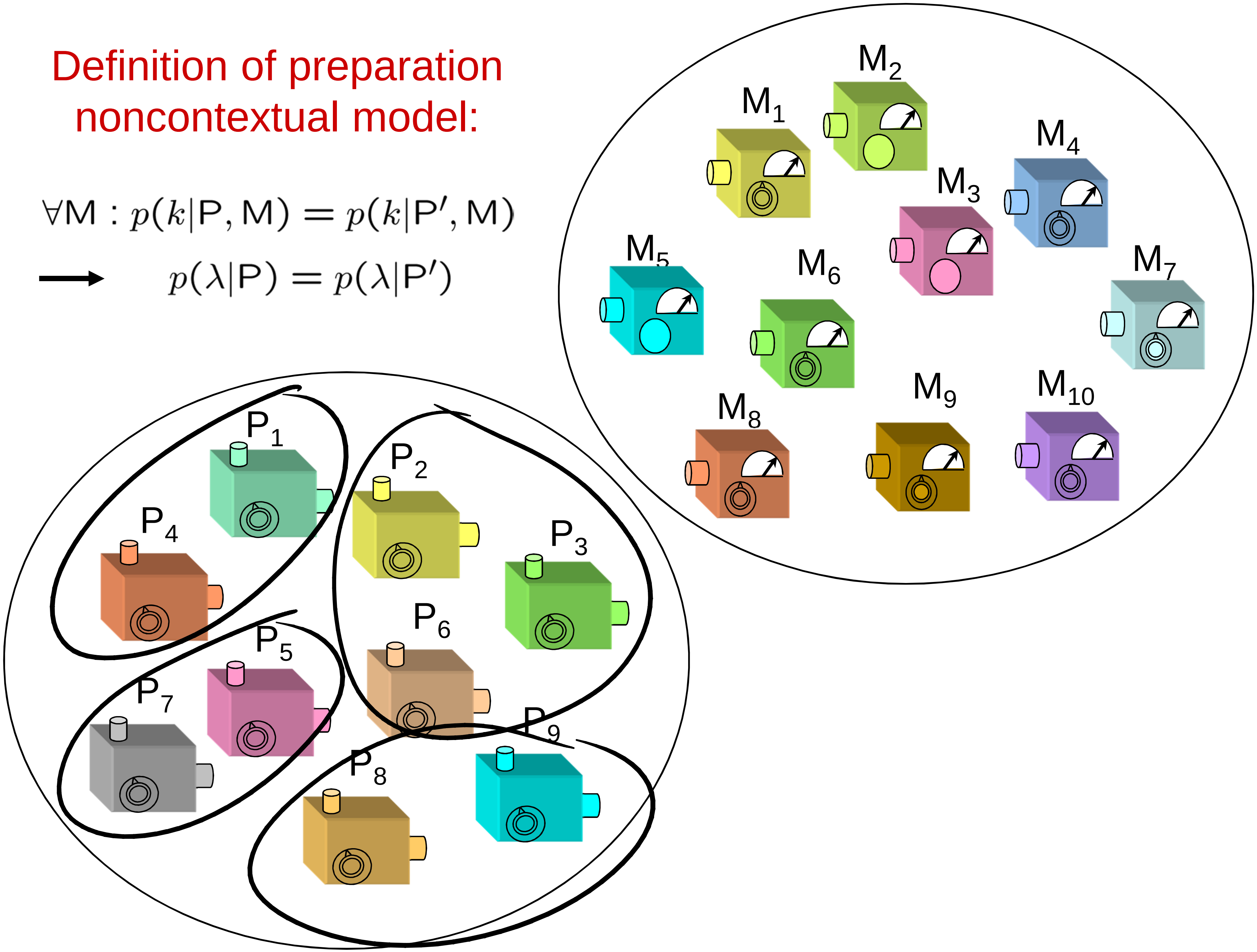
Preparation contextual model

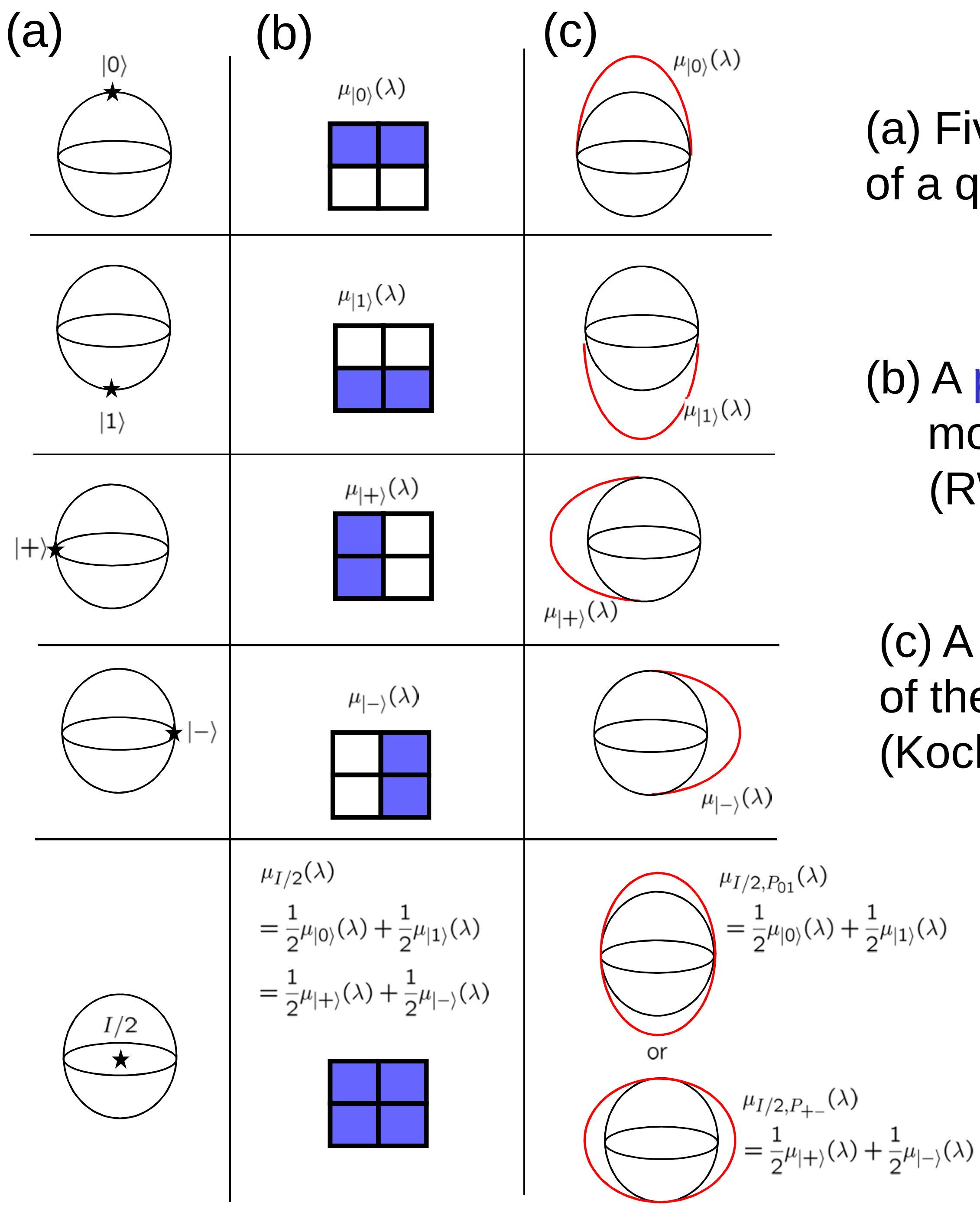


Definition of preparation noncontextual model:

$$\forall \mathsf{M} : p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P}',\mathsf{M})$$

$$\longrightarrow \quad p(\lambda|\mathsf{P}) = p(\lambda|\mathsf{P}')$$





(a) Five operational states of a qubit

(b) A preparation noncontextual model of these (RWS, 2005)

(c) A preparation contextual model of these (Kochen-Specker, 1967)

Universal noncontextuality = measurement noncontextuality and preparation noncontextuality

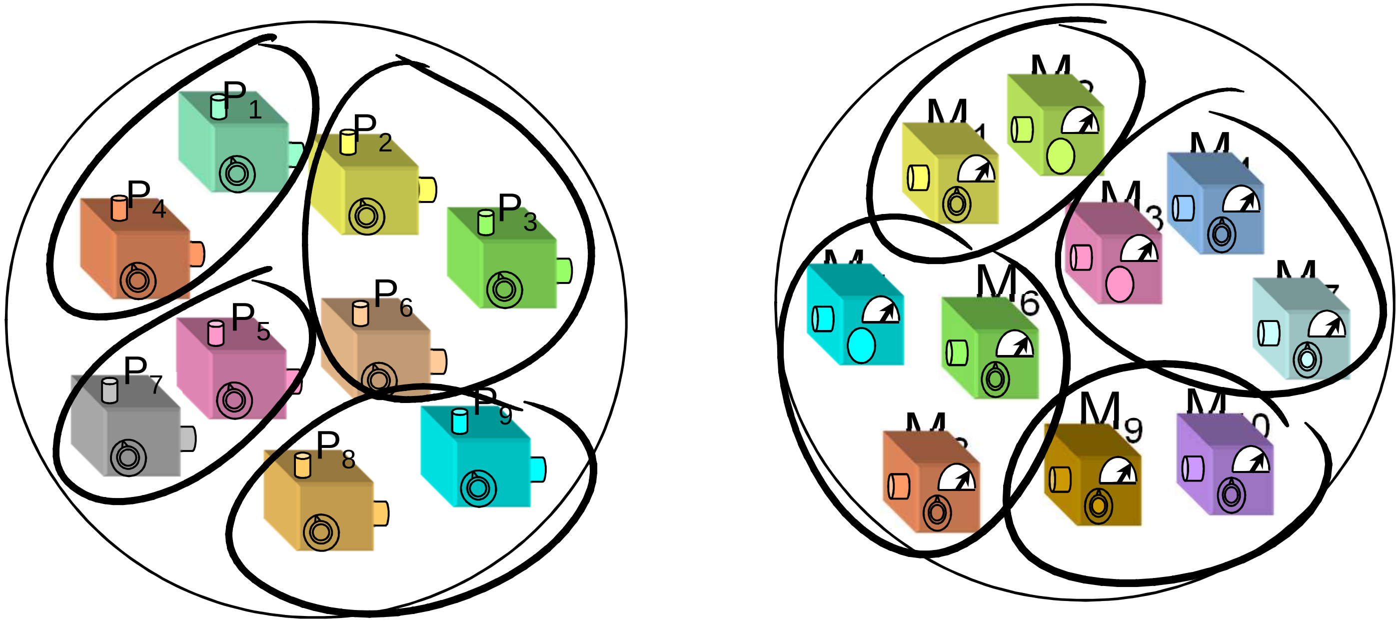
Claim: Preparation noncontextuality is as natural (or unnatural) as measurement noncontextuality

Q: Why is noncontextuality plausible at all?

A: This methodological principle: if a difference in set-up is not distinguished in the observable phenomena then it should not be distinguished in the ontological picture either

Quantum theory does not admit of a universally noncontextual hidden variable model

For an arbitrary operational theory, when is a universally noncontextual HV model possible?



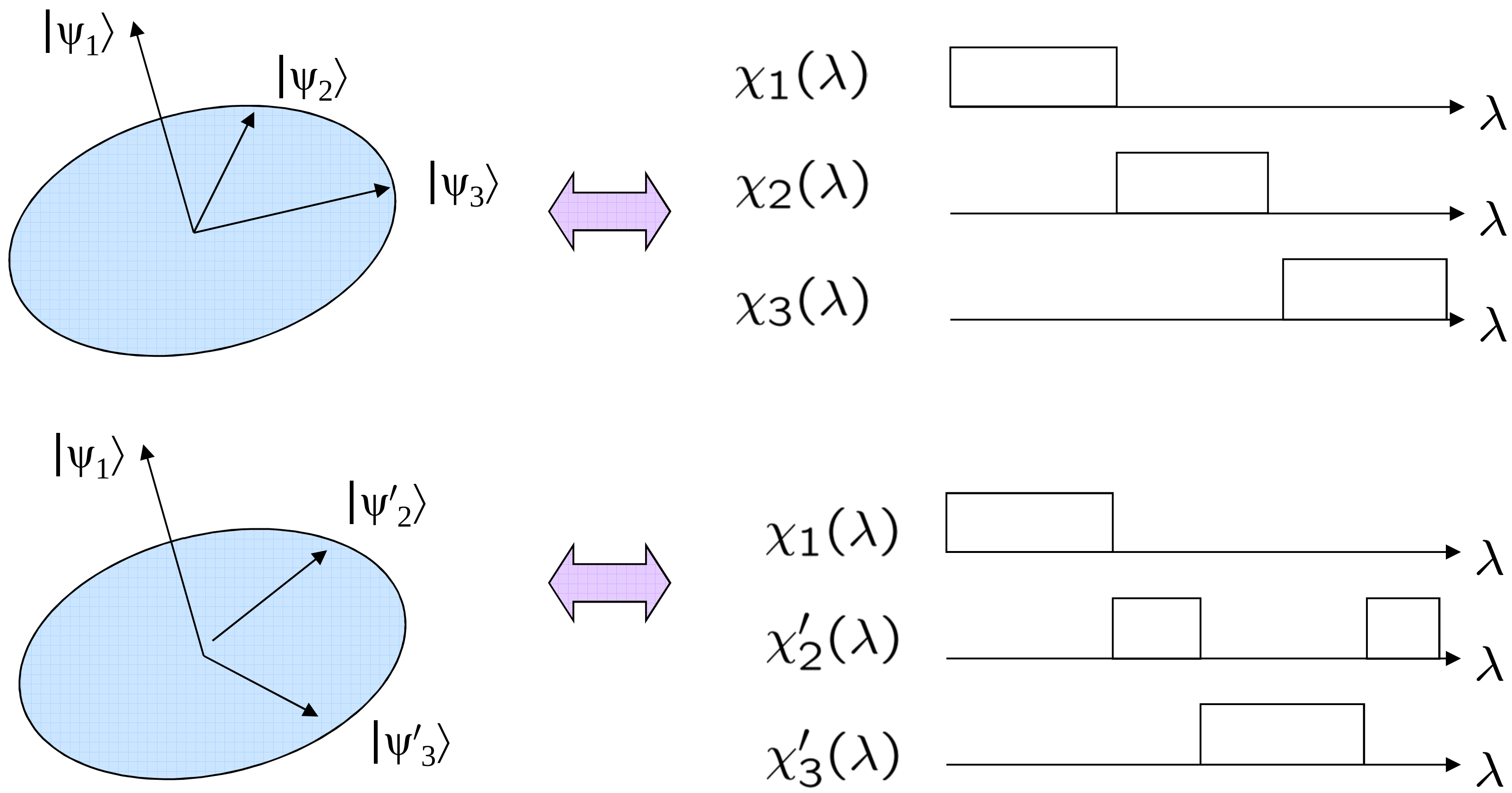
A set of preparations and measurements defines statistics $p(k|\mathrm{P_i},\mathrm{M_j})$

Assumption of a universally noncontextual HV model
→ ***NONCONTEXTUALITY INEQUALITIES***

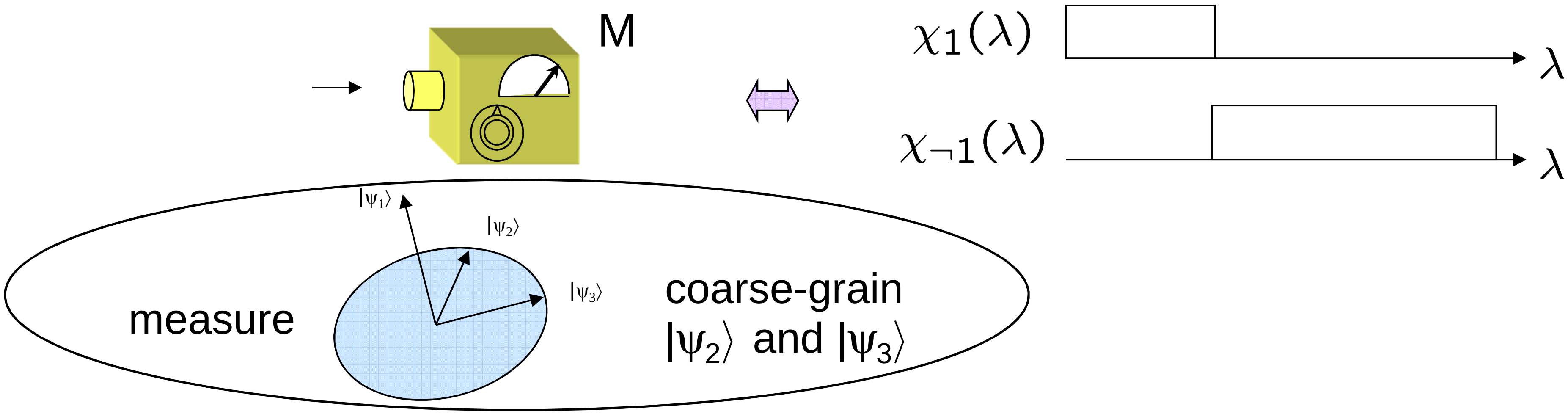
$$F(p(k_1|\mathrm{P}_1,\mathrm{M}_1),p(k_1|\mathrm{P}_2,\mathrm{M}_1),p(k_2|\mathrm{P}_1,\mathrm{M}_2),\ \ldots) \leq \text{bound}$$

Traditional noncontextuality
versus
Measurement noncontextuality

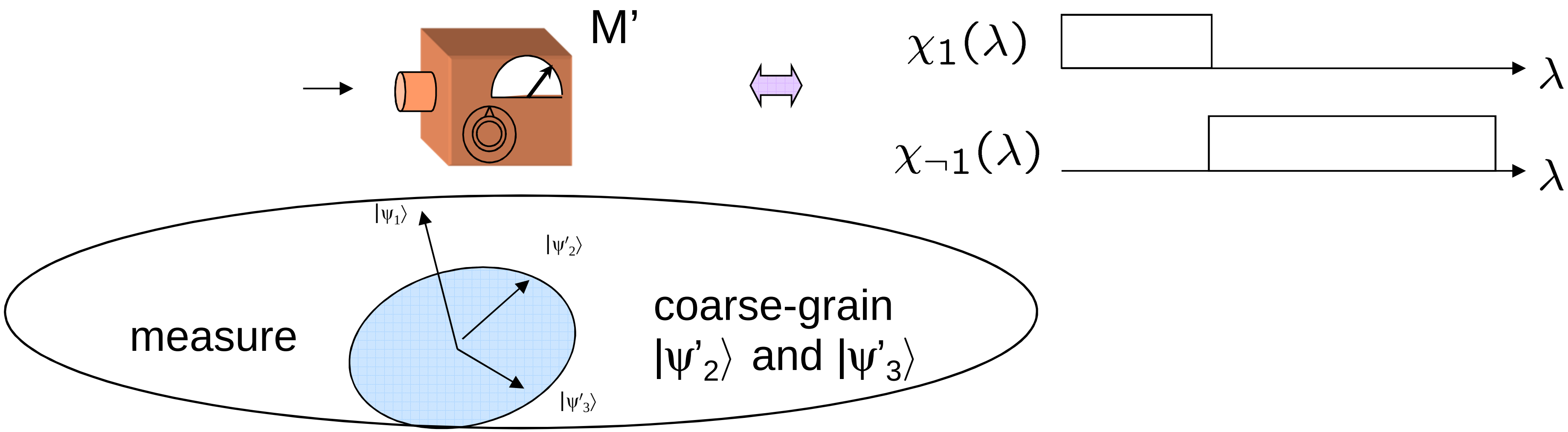
The traditional notion of noncontextuality:



This is equivalent to assuming:

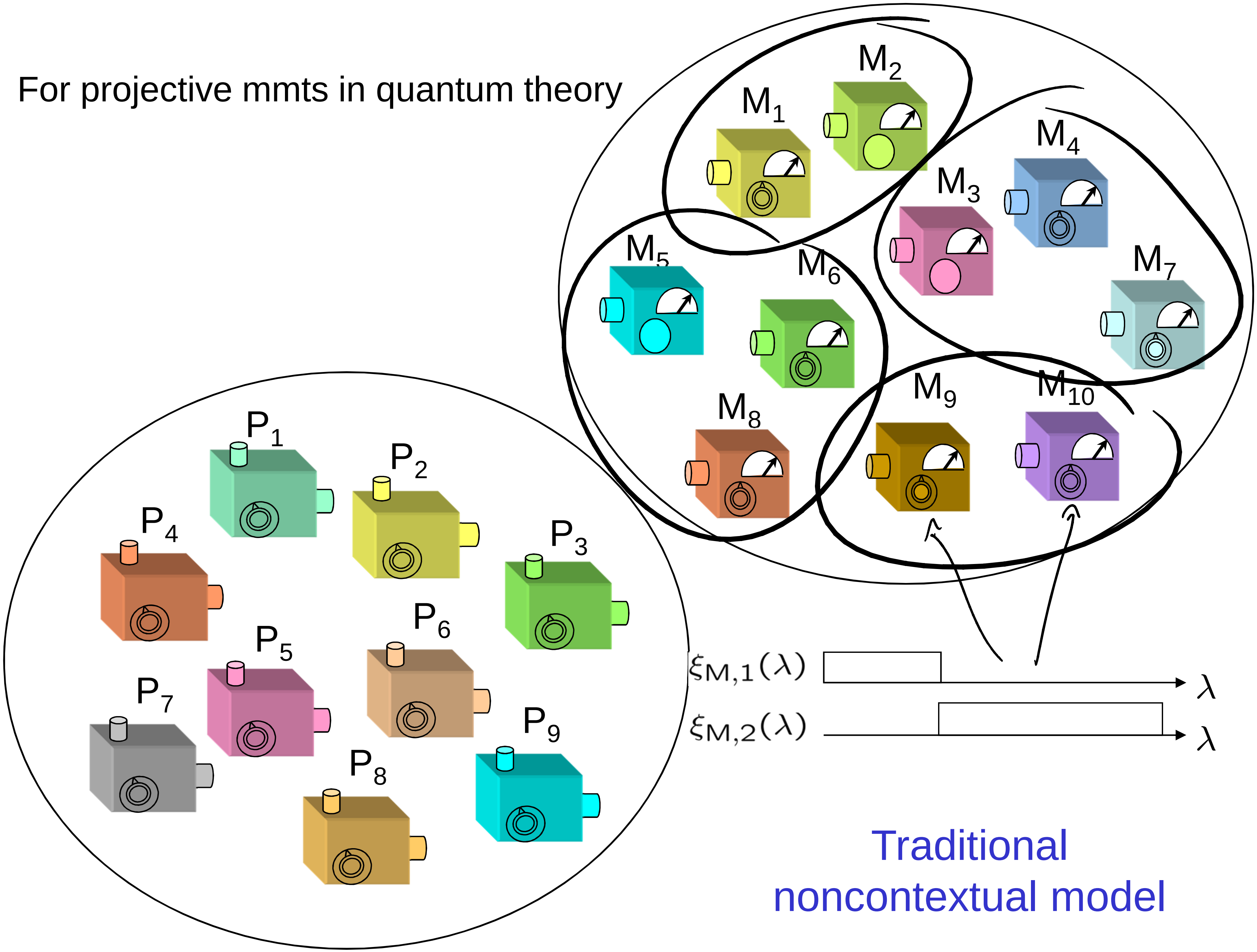


$$\{|\psi_1\rangle\langle\psi_1|,\ I - |\psi_1\rangle\langle\psi_1|\}$$



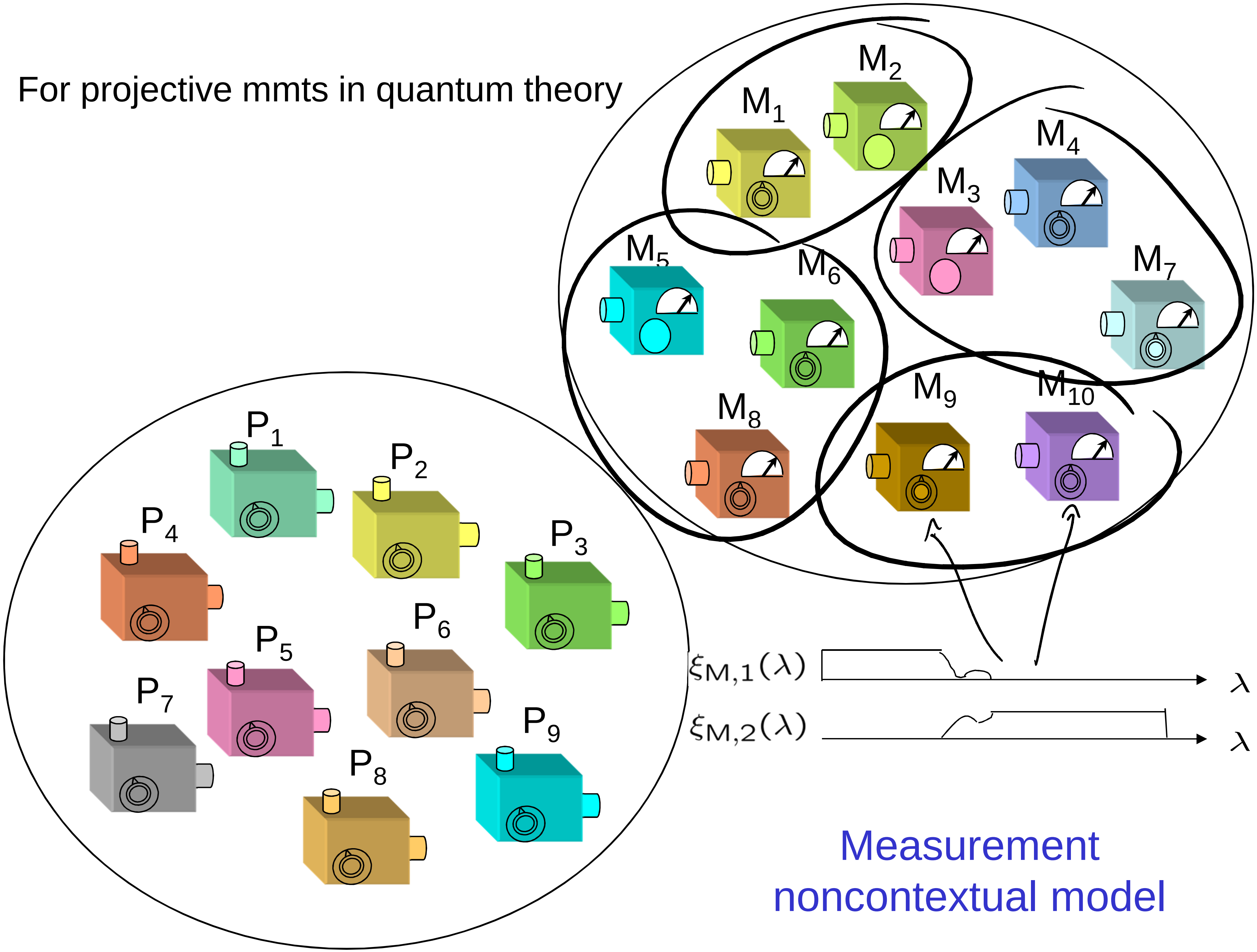
$$\{|\psi_1\rangle\langle\psi_1|,\ I - |\psi_1\rangle\langle\psi_1|\}$$

For projective mmts in quantum theory



Traditional noncontextual model

For projective mmts in quantum theory



Measurement noncontextual model

So, while measurement noncontextuality is
a generalization of the traditional notion

(from projective to nonprojective measurements
and from HV models of quantum theory
to HV models of any operational theory)

For projective measurements, it is actually a departure
from the traditional notion

Local determinism:

We ask: Does the outcome depend on space-like separated events (in addition to local settings and λ)?

Local causality:

We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and λ)?

Traditional notion of noncontextuality:

We ask: Does the outcome depend on the measurement context (in addition to the observable and λ)?

The revised notion of measurement noncontextuality:

We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and λ)?

Noncontextuality and determinism are separate issues measurement noncontextuality

and

outcome determinism
for projective measurements

→

traditional notion of
noncontextuality

No-go theorems for previous notion are not necessarily
no-go theorems for the new notion!

In face of contradiction, we could give up outcome determinism

Can we justify the assumption of outcome determinism?

Many people have a strong intuition that allowing outcome *in*determinism does not add any generality and that consequently we may as well assume outcome determinism.

A (flawed) argument in favour of outcome determinism

Premiss: Every measurement can be represented by an outcome-deterministic response function on a larger system

"Neumark extension" at hidden variable level

Premiss: If two measurements have the same statistics for all preparations, then they should be represented by identical response functions in the hidden variable model

Measurement noncontextuality

Purported conclusion: If two measurements have the same statistics for all preparations, then they should be represented by identical outcome-deterministic response functions

Exhibiting the flaw

Premiss: Every measurement on *s* can be represented by an outcome-deterministic response function on *sa* together with a distribution on *a*

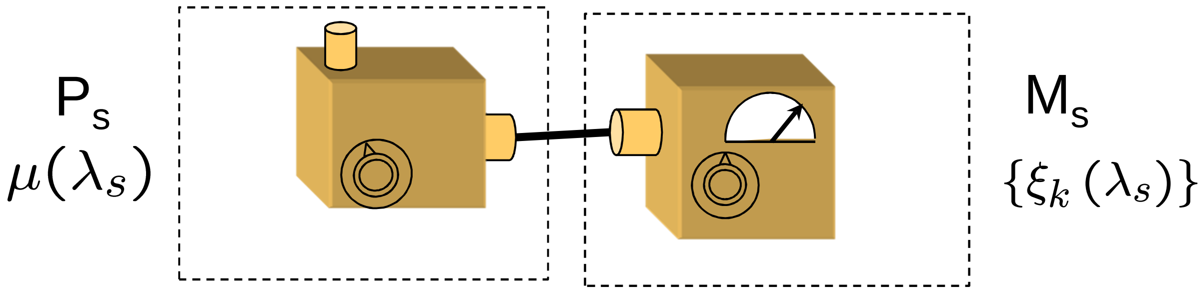
"Neumark extension" at hidden variable level

Premiss: If two measurements on *s* have the same statistics for all preparations on *s*, then they should be represented by identical response functions on *s*

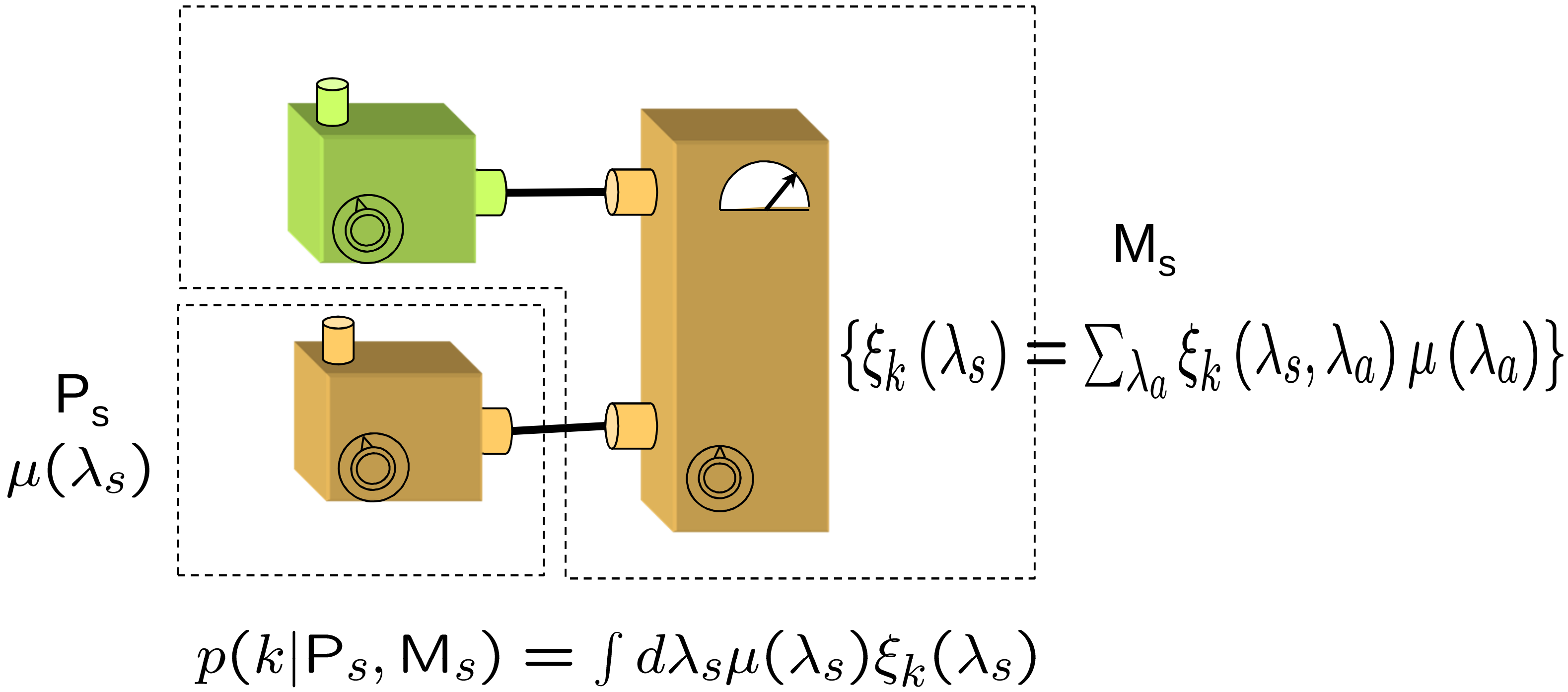
Measurement noncontextuality

Purported conclusion: If two measurements on *s* have the same statistics for all preparations on *s*, then they should be represented by identical outcome-deterministic response functions *on sa*.

But there is **no reason to think they are identical on *sa***



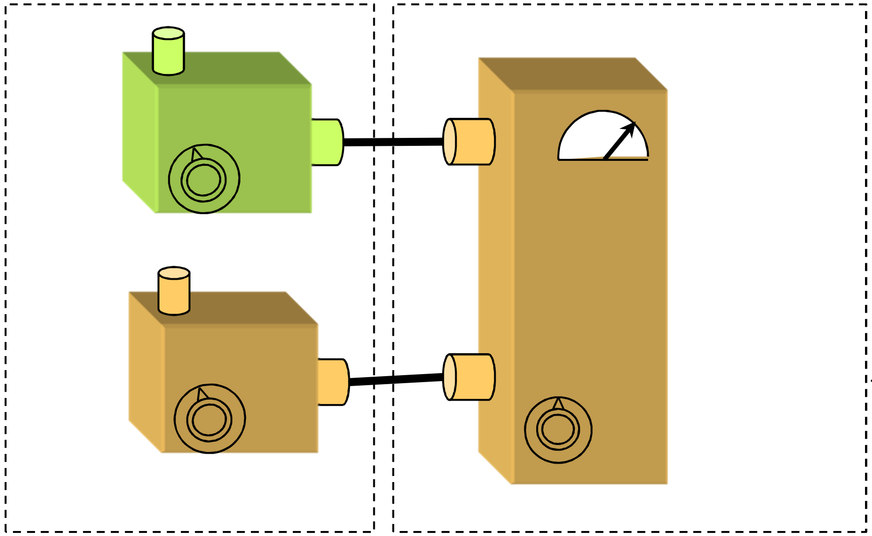
$$p(k|\mathsf{P}_s, \mathsf{M}_s) = \int d\lambda_s \mu(\lambda_s) \xi_k(\lambda_s)$$



M_s
$\{\xi_k(\lambda_s) = \sum_{\lambda_a} \xi_k(\lambda_s, \lambda_a)\mu(\lambda_a)\}$
P_s
$\mu(\lambda_s)$
$p(k|\mathsf{P}_s, \mathsf{M}_s) = \int d\lambda_s \mu(\lambda_s)\xi_k(\lambda_s)$

$P_{sa} = (P_s, P_a)$

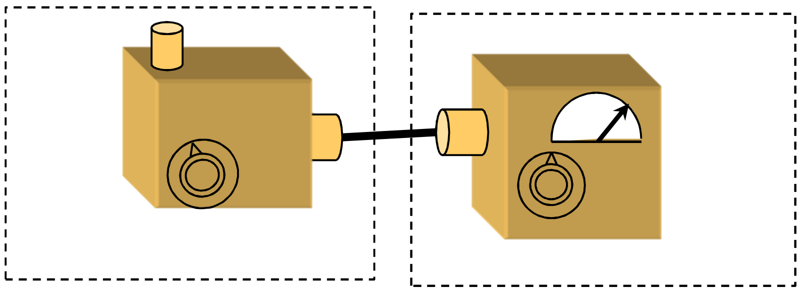
$\mu(\lambda_s, \lambda_a)$
$= \mu(\lambda_s)\mu(\lambda_a)$



M_{sa}

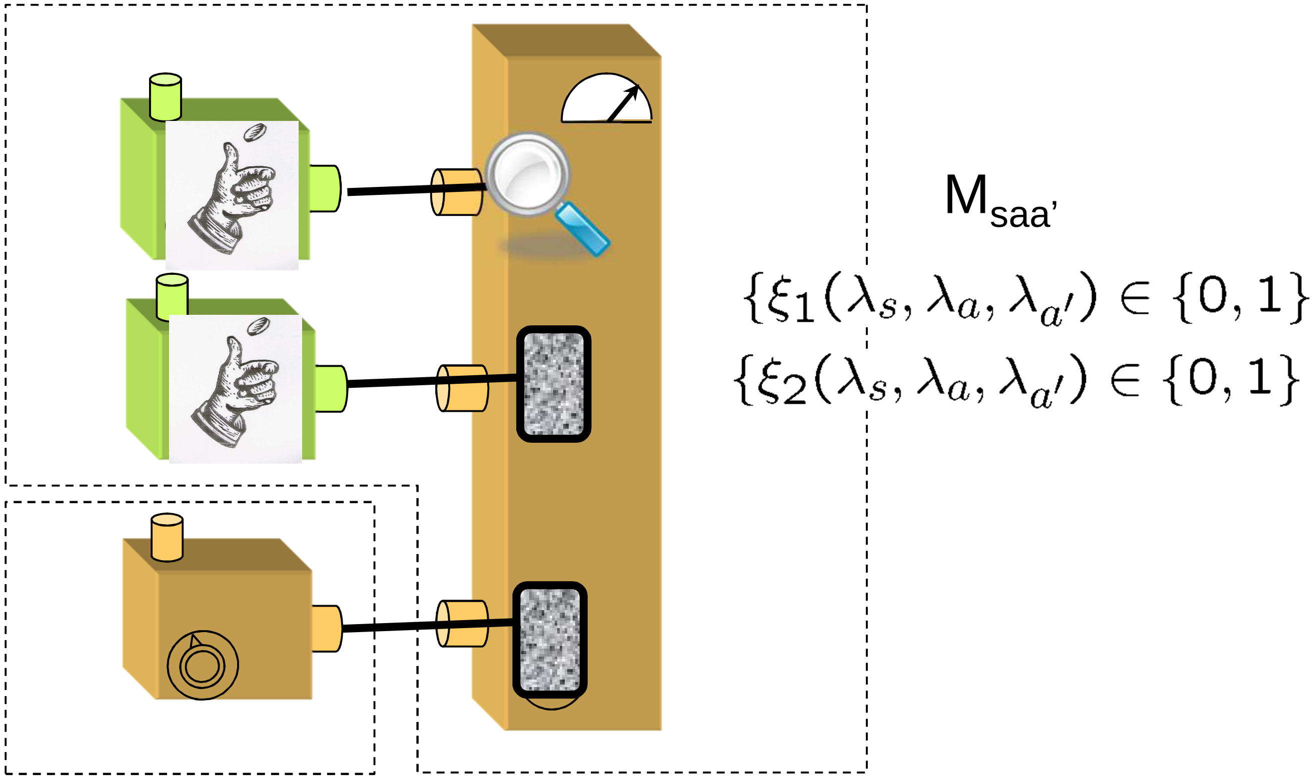
$\{\xi_k(\lambda_s, \lambda_a)\}$

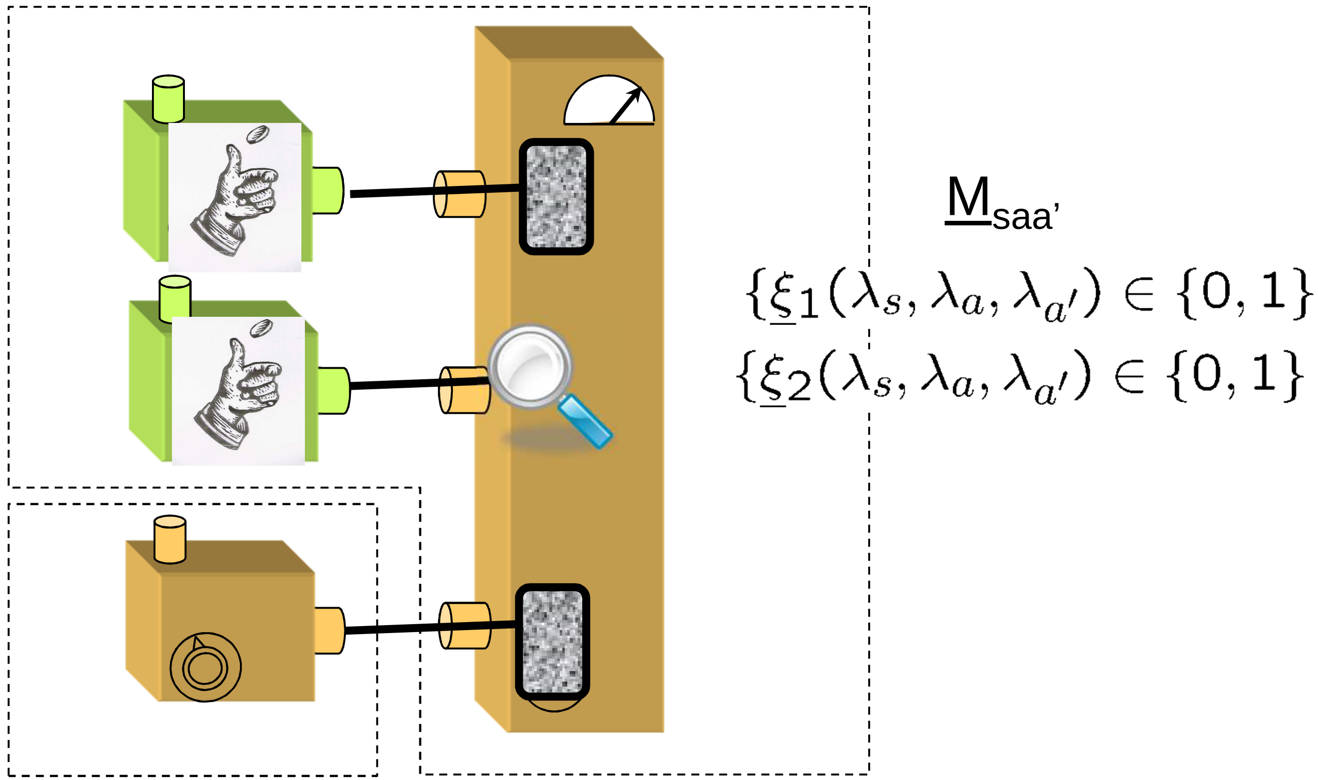
$$p(k|\mathsf{P}_{sa}, \mathsf{M}_{sa}) = \int d\lambda_s d\lambda_a \mu(\lambda_s, \lambda_a)\xi_k(\lambda_s, \lambda_a)$$



$\mathrm{M_s}$

$$\xi_1(\lambda_s) = \frac{1}{2} \quad \forall \lambda_s$$
$$\xi_2(\lambda_s) = \frac{1}{2} \quad \forall \lambda_s$$


$M_{saa'}$
$\{\xi_1(\lambda_s, \lambda_a, \lambda_{a'}) \in \{0, 1\}$
$\{\xi_2(\lambda_s, \lambda_a, \lambda_{a'}) \in \{0, 1\}$


$\underline{M}_{saa'}$
$\{\underline{\xi}_1(\lambda_s, \lambda_a, \lambda_{a'}) \in \{0, 1\}$
$\{\underline{\xi}_2(\lambda_s, \lambda_a, \lambda_{a'}) \in \{0, 1\}$

Can we justify the assumption of outcome determinism?

A qualified “Yes”

- for projective measurements only
- assuming the validity of quantum theory
- The proof appeals to preparation noncontextuality

Recall:

measurement noncontextuality

and

outcome determinism
for projective measurements

→ traditional notion of noncontextuality

No-go theorems for traditional notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up outcome determinism

Assuming the validity of quantum theory, one can prove that

preparation noncontextuality $\longrightarrow$ outcome determinism for projective measurements

And therefore:

measurement noncontextuality

and

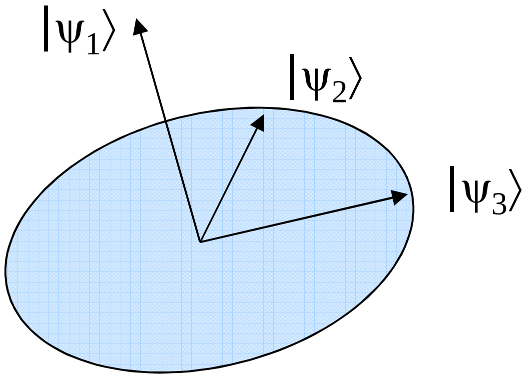
preparation noncontextuality

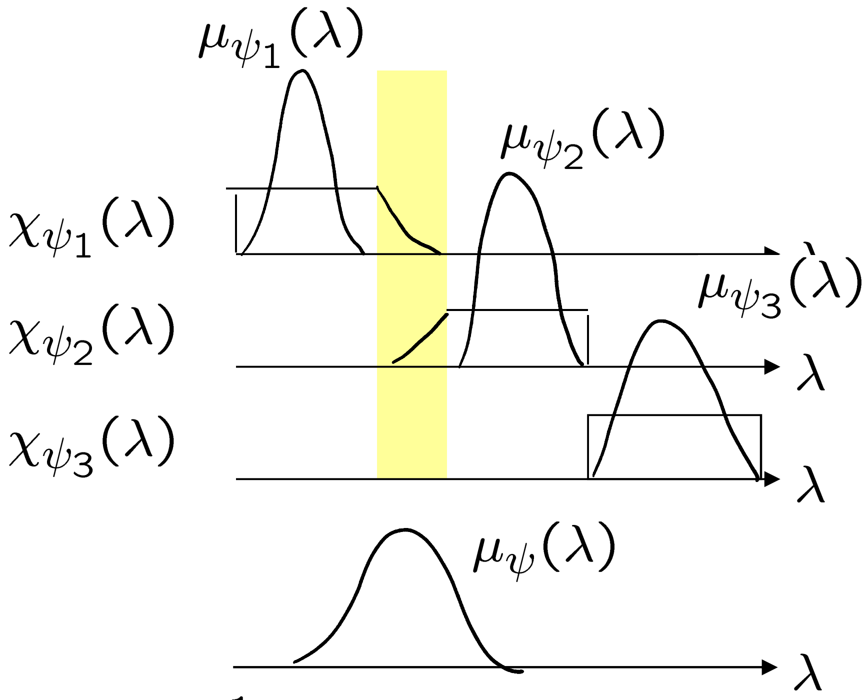
$\longrightarrow$ Traditional notion of noncontextuality

Assuming the validity of quantum theory, one can prove that

preparation noncontextuality → outcome determinism for projective measurements

Proof





$$\mu_{I/3}(\lambda) = \tfrac{1}{3}\mu_{\psi_1}(\lambda) + \tfrac{1}{3}\mu_{\psi_2}(\lambda) + \tfrac{1}{3}\mu_{\psi_3}(\lambda)$$

$$\mu_{I/3}(\lambda) = \tfrac{1}{3}\mu_{\psi}(\lambda) + \ldots$$

Assuming the validity of quantum theory, one can prove that

preparation noncontextuality → outcome determinism for projective measurements

And therefore:

measurement noncontextuality
and
preparation noncontextuality → Traditional notion of noncontextuality

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion

… and there are many new proofs

However, what is needed for a measurement-based experimental test of contextuality is:

- An operational notion of sharp measurement (corresponding to a projective measurement in quantum theory)
- A justification of outcome determinism for these

preparation noncontextuality

and

? Experimental facts about sharp measurements in a general operational theory ?

→

outcome determinism for sharp measurements in a general operational theory

Operational test of a noncontextuality inequality and its experimental violation

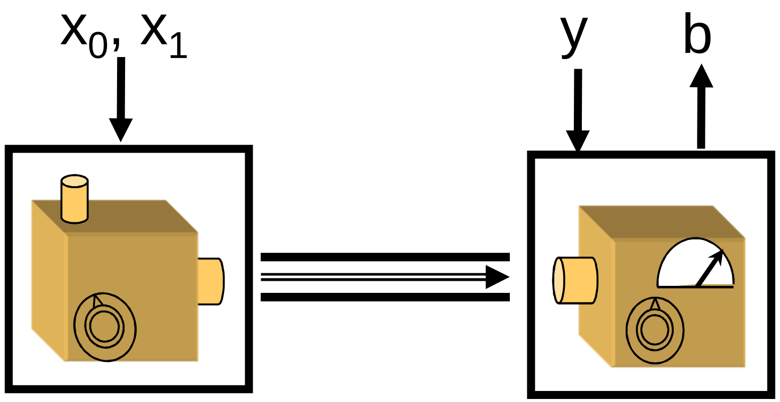
(almost) independent of the validity of quantum theory

Buzacott, Keehn, Pryde, Toner, RWS, PRL 102, 010401 (2009)
Inspired by thesis work of Ernesto Galvao

The game of parity-oblivious multiplexing

Victor

Alice

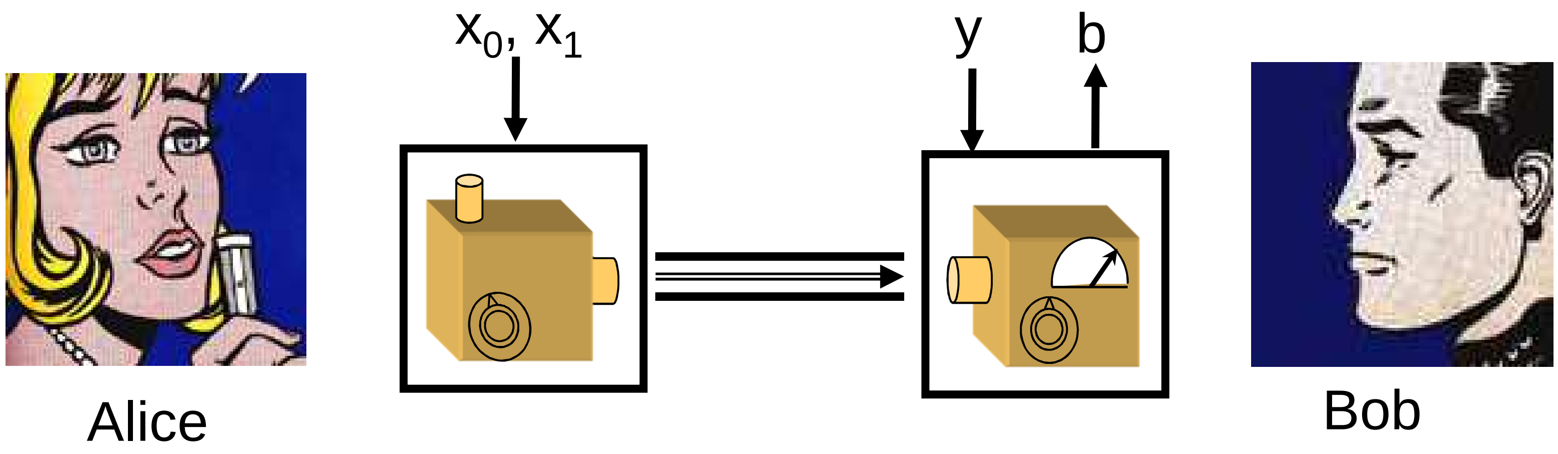


Bob

Alice and Bob win if

$b = x_y$

The catch: no information about parity ($x_0 \oplus x_1$) can be conveyed!



The classical world

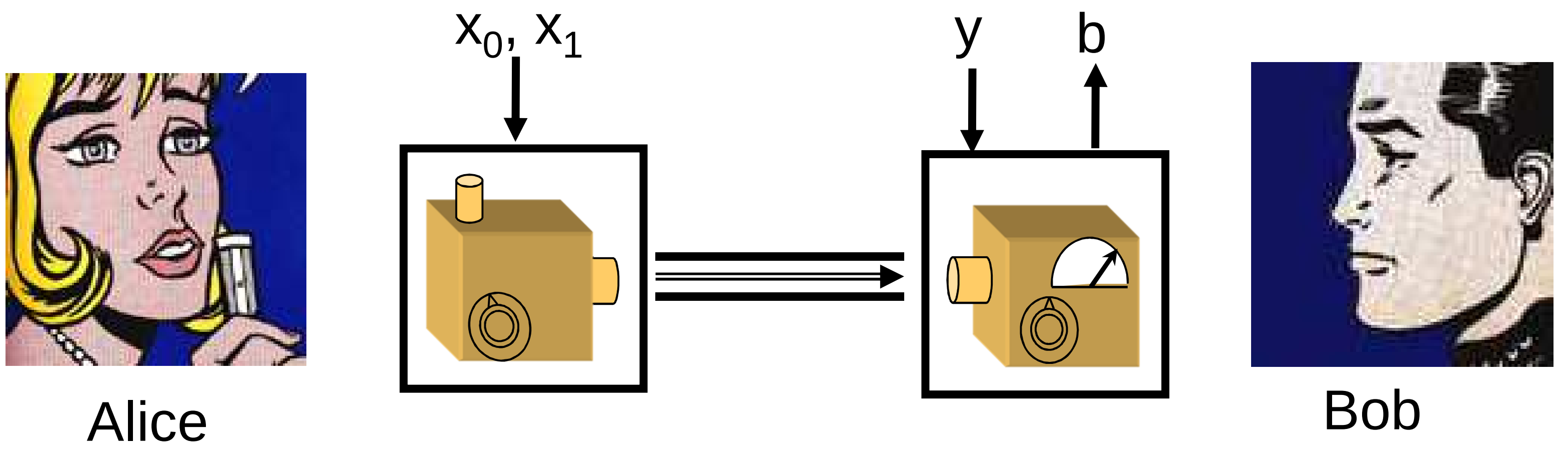
Deterministic strategies

Any function depending on *both* x_0 and x_1 reveals info about $x_0 \oplus x_1$

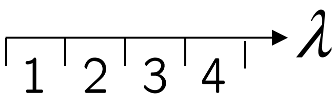
An optimal protocol: she always sends x_0 (and Bob knows this)

Optimal probability of success: $\frac{1}{2}(1) + \frac{1}{2}(\frac{1}{2}) = 3/4$

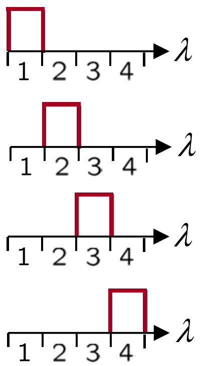
$$p(b=x_y) = 3/4$$



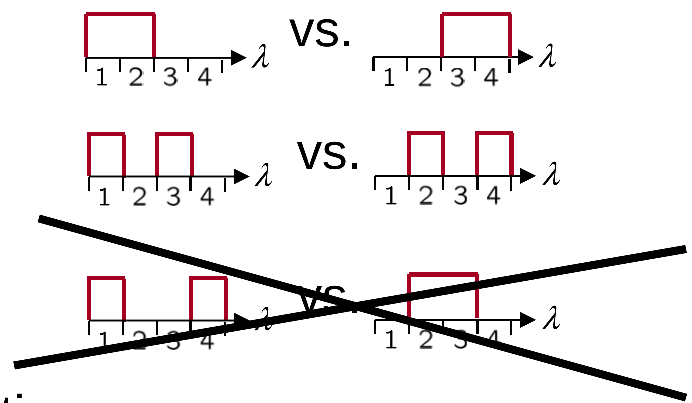
An imaginary world



extremal preparations



measurements

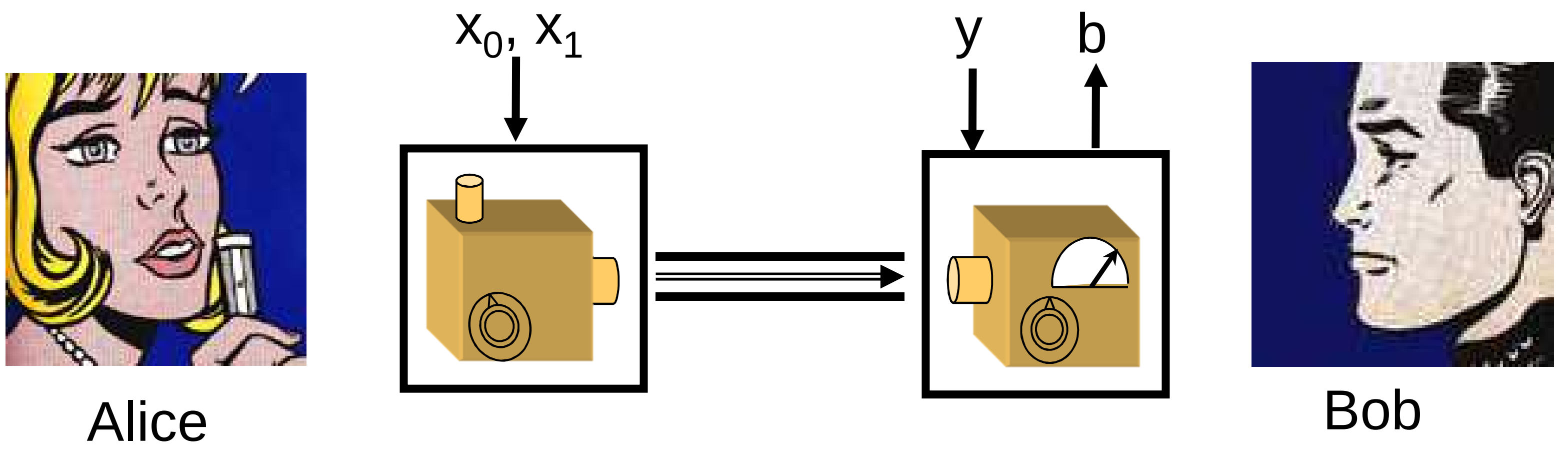


Protocol: Alice encodes x_0, x_1 into preparation

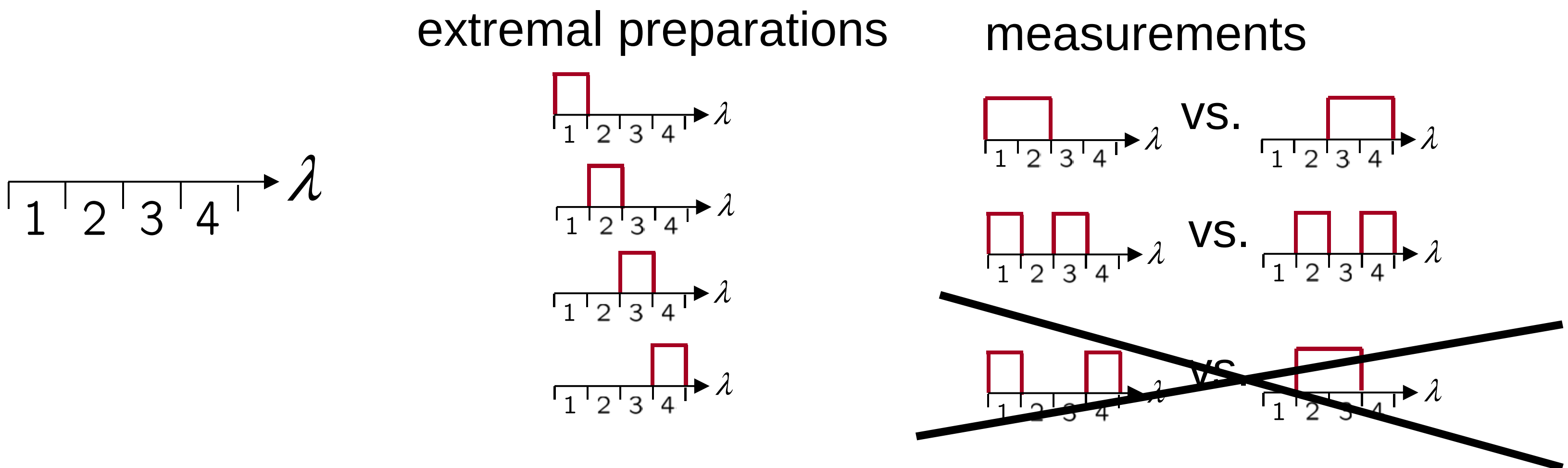
Bob measures x_y

No measurement can reveal anything about $x_0 \oplus x_1$

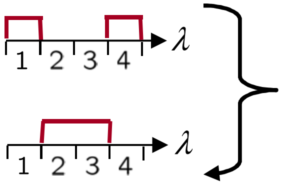
$$p(b=x_y) = 1$$



An imaginary world

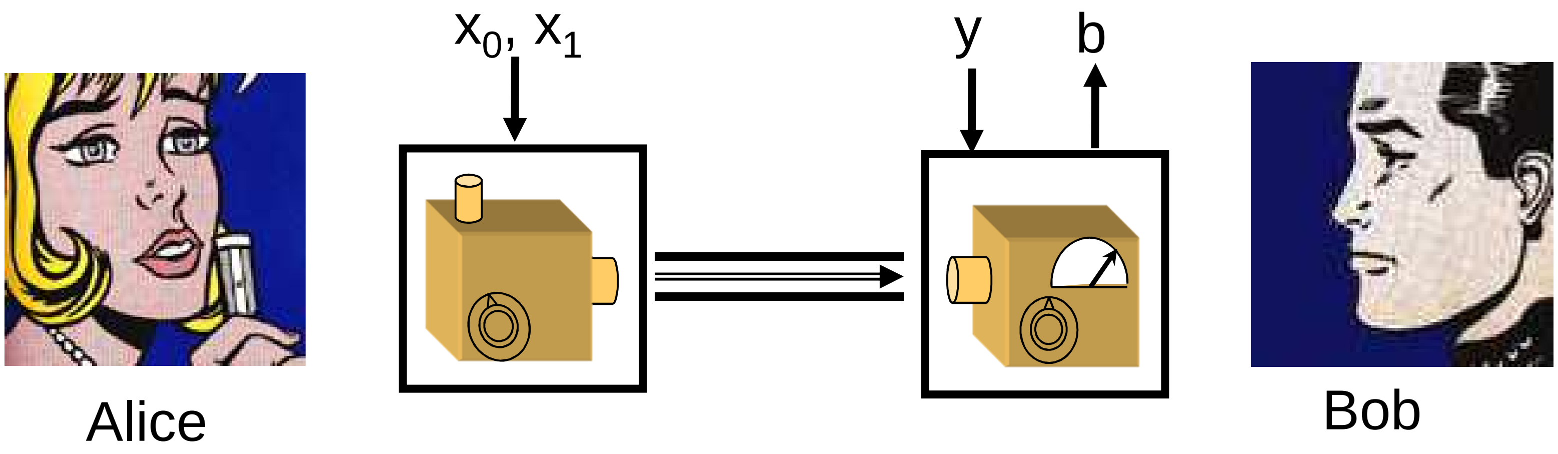


Note: for non-extremal preparations

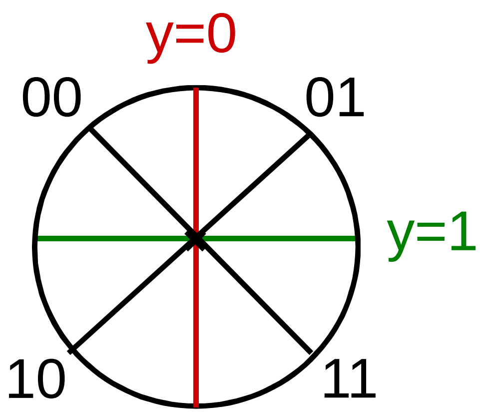


Indistinguishable at operational level
Distinguishable at hidden variable level

This world is preparation contextual



The quantum world



Wiesner's multiplexing scheme

$$p(b=x_y) \simeq 0.8536$$

And it's parity-oblivious

Wiesner, SIGACT News 15, 78 (1983).
Ambainis, Nayak, Ta-Shma, Vazirani, in Proc. 31st Annual ACM Symposium on the Theory of Computing (1999).

What will be shown:

Theorem: For all operational theories admitting a preparation noncontextual model

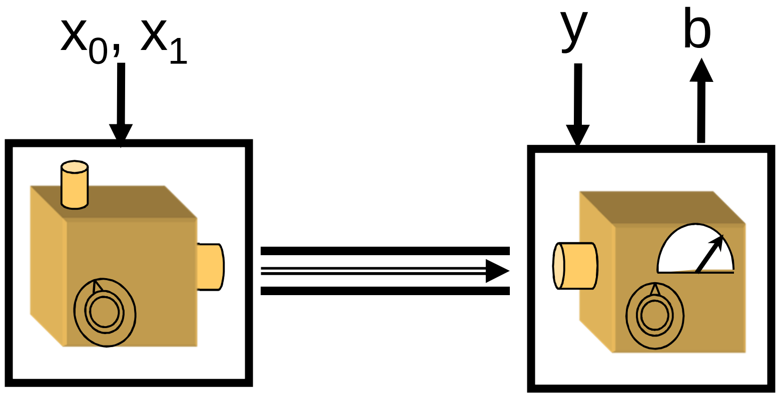
$$p(b=x_y) \leq 3/4$$

A "noncontextuality inequality"

Derivation of the noncontextuality inequality

P_{x_0,x_1}

Implement one of P_{00}, P_{01}, P_{10}, P_{11}



M_y

Implement one of M_0, M_1 (report outcome 0 or 1)

$P_{x_0 \oplus x_1 = 0}$ = P_{00} with prob. ½, P_{11} with prob. ½

$P_{x_0 \oplus x_1 = 1}$ = P_{01} with prob. ½, P_{10} with prob. ½

$$\forall M \forall k : \; p(k|M, P_{x_0 \oplus x_1 = 0}) = p(k|M, P_{x_0 \oplus x_1 = 1})$$ Parity-oblivious

By preparation noncontextuality

⇩

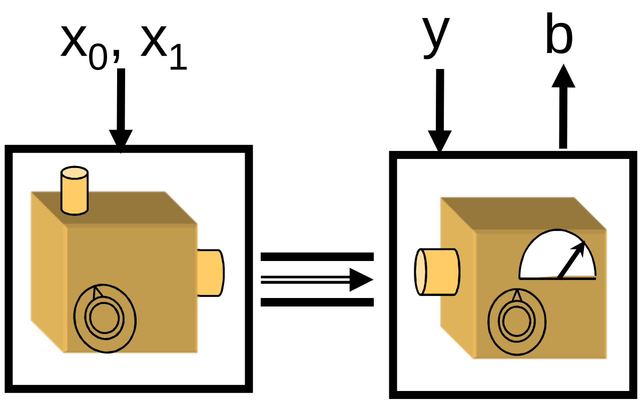
$$p(\lambda|P_{x_0 \oplus x_1 = 0}) = p(\lambda|P_{x_0 \oplus x_1 = 1})$$

$$p(P_{x_0 \oplus x_1 = 0}|\lambda) = p(P_{x_0 \oplus x_1 = 1}|\lambda)$$

So λ satisfies the same constraint as a classical message

$$p(b = x_y) \leq 3/4$$

Experimental test of noncontextuality inequality

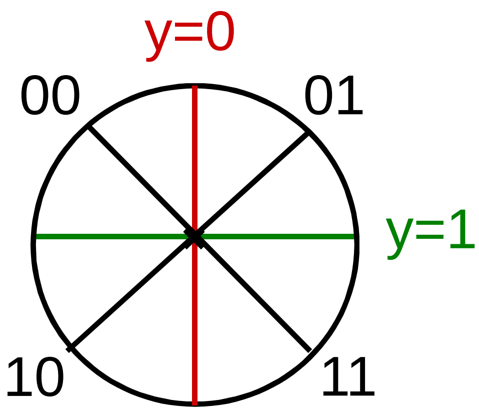


Measure $p(k|M_y, P_{x_0 x_1})$ calculate $p(b = x_y)$

Verify $p(b = x_y) > \frac{3}{4}$

Verify parity-oblivious property

$$\forall M : \; p(k|M, P_{x_0 \oplus x_1 = 0}) = p(k|M, P_{x_0 \oplus x_1 = 1})$$



This noncontextuality inequality is violated experimentally

What is mysterious about contextuality?

Noncontextuality inequality violations
(i.e. dependence of distributions on preparation context)

tension

No way to use the context-dependence to communicate information
(i.e. independence of measurement statistics on preparation context)

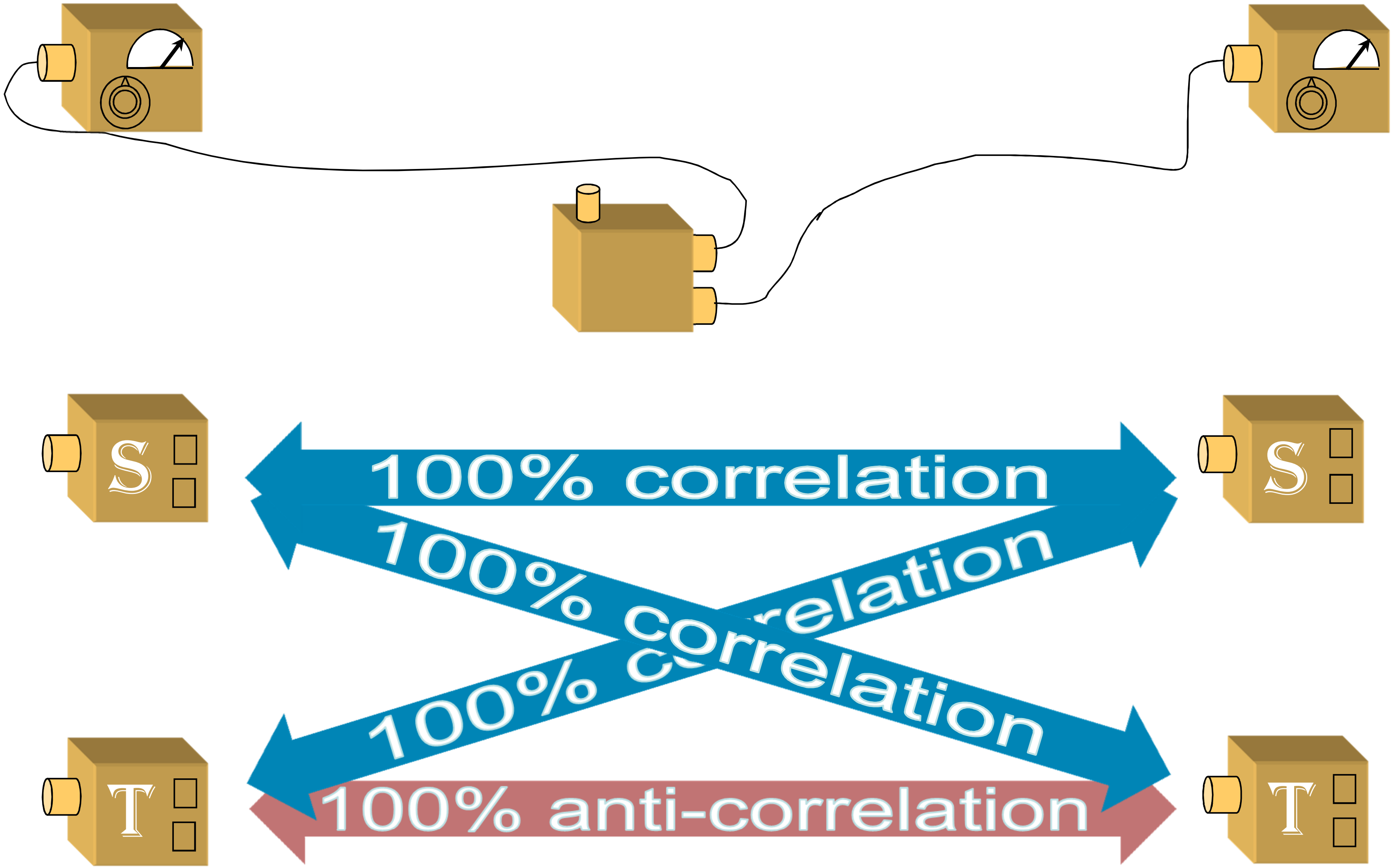
Compare with what is mysterious about nonlocality

Bell inequality violations
(i.e. there must be superluminal causation)

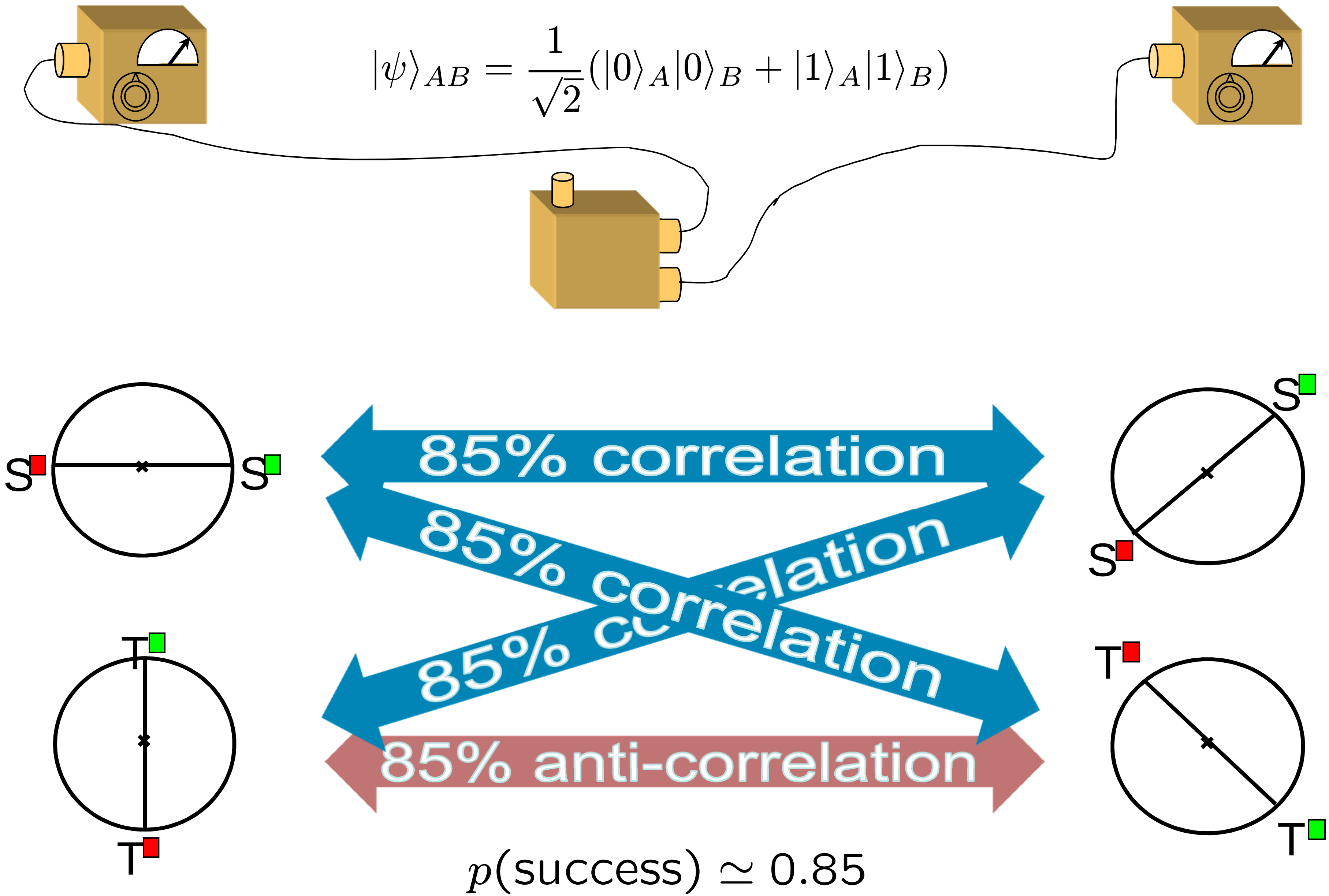
tension

No superluminal signalling
(i.e. there is no way to make use of the superluminal causation)

Connection between preparation contextuality and nonlocality

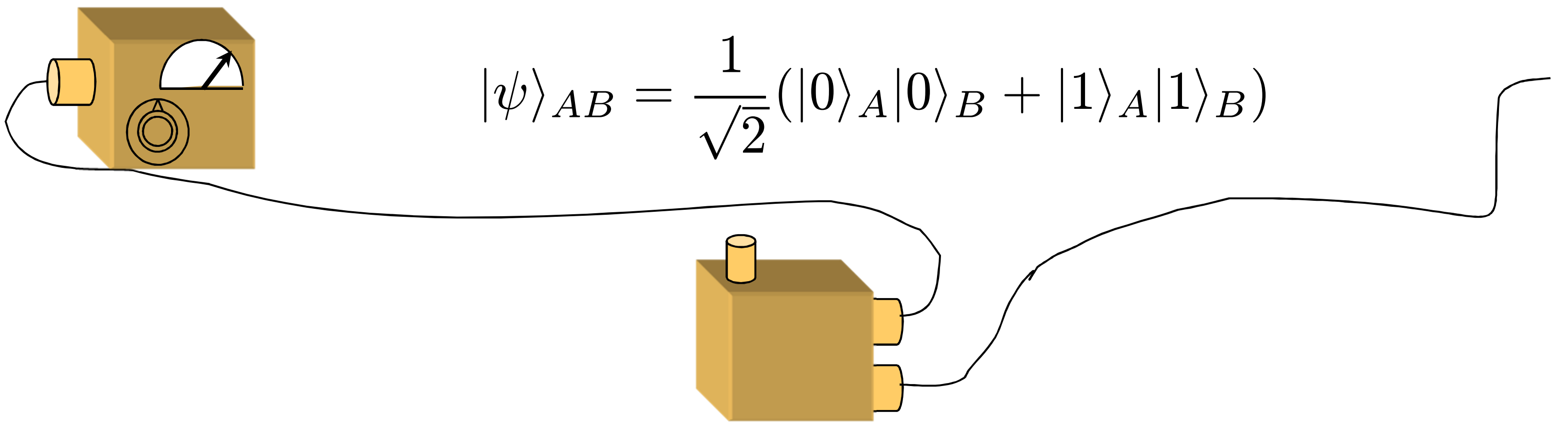


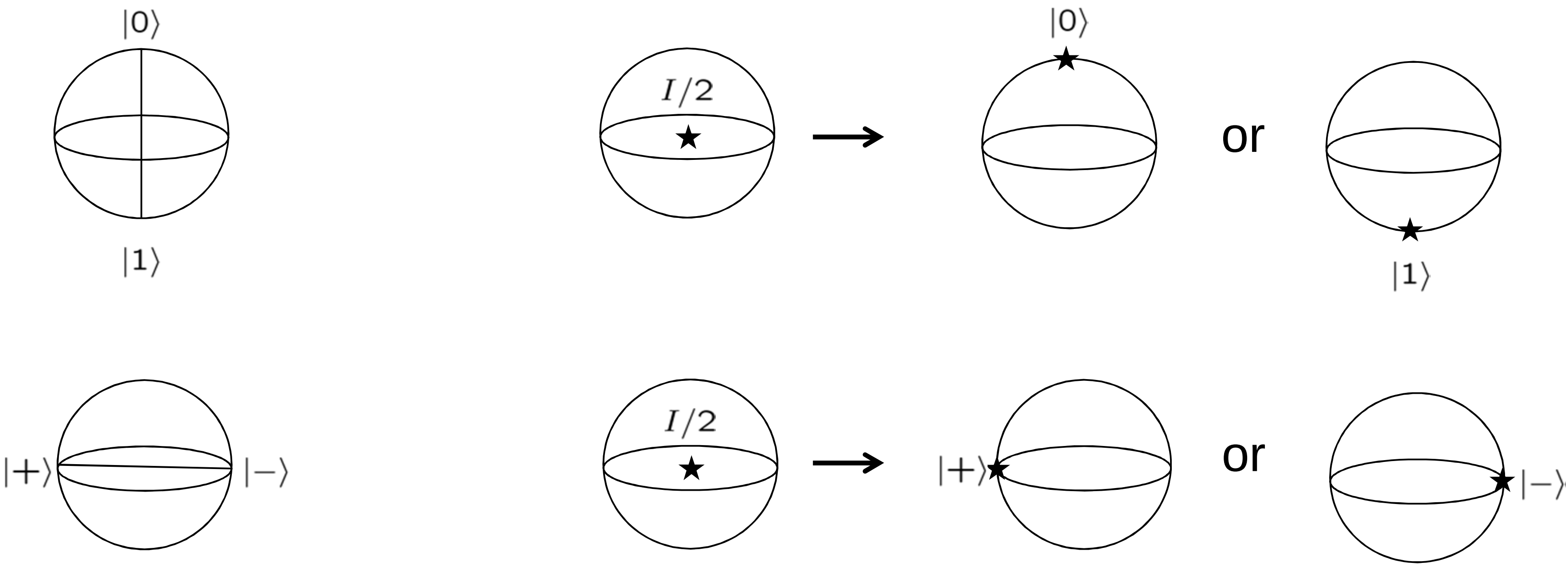
$p(\text{success}) \leq 0.75$

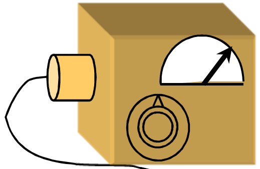


$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$
S
S
S
S
T
T
T
T
85% correlation
85% correlation
85% correlation
85% anti-correlation
$p(\text{success}) \simeq 0.85$

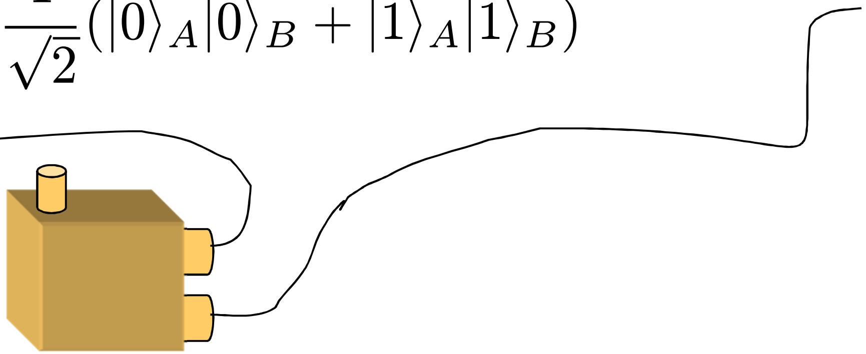
$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$







$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$



$\{|0\rangle_A, |1\rangle_A\}$

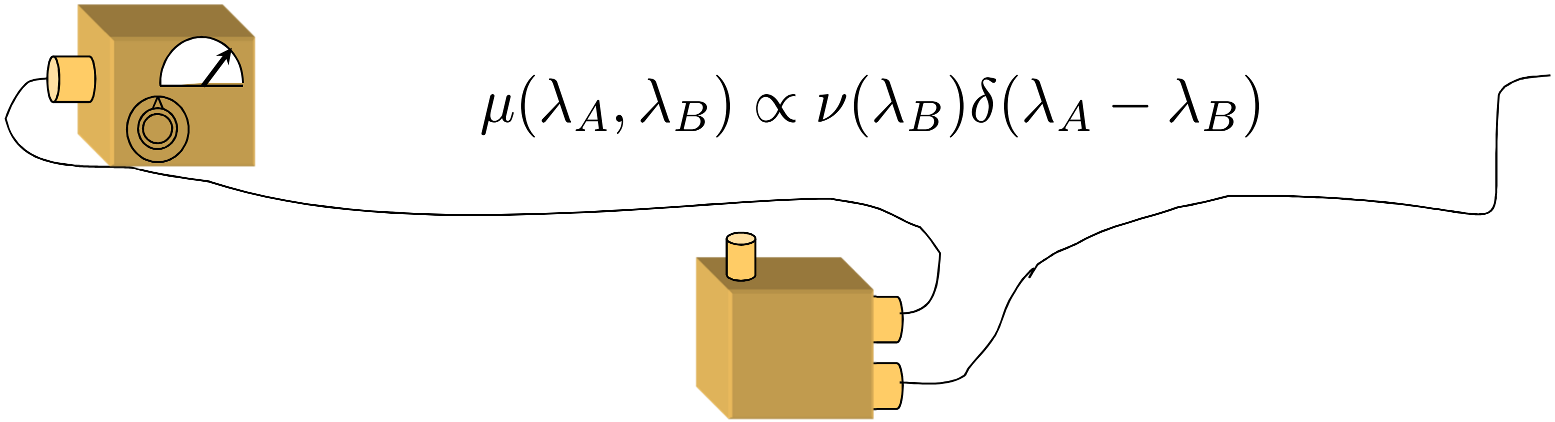
$$I_B/2 \longrightarrow |0\rangle_B \quad \text{or} \quad |1\rangle_B$$

where $$\frac{1}{2}|0\rangle_B\langle 0| + \frac{1}{2}|1\rangle_B\langle 1| = I_B/2$$

$\{|+\rangle_A, |-\rangle_A\}$

$$I_B/2 \longrightarrow |+\rangle_B \quad \text{or} \quad |-\rangle_B$$

where $$\frac{1}{2}|+\rangle_B\langle +| + \frac{1}{2}|-\rangle_B\langle -| = I_B/2$$



$$\mu(\lambda_A, \lambda_B) \propto \nu(\lambda_B)\delta(\lambda_A - \lambda_B)$$

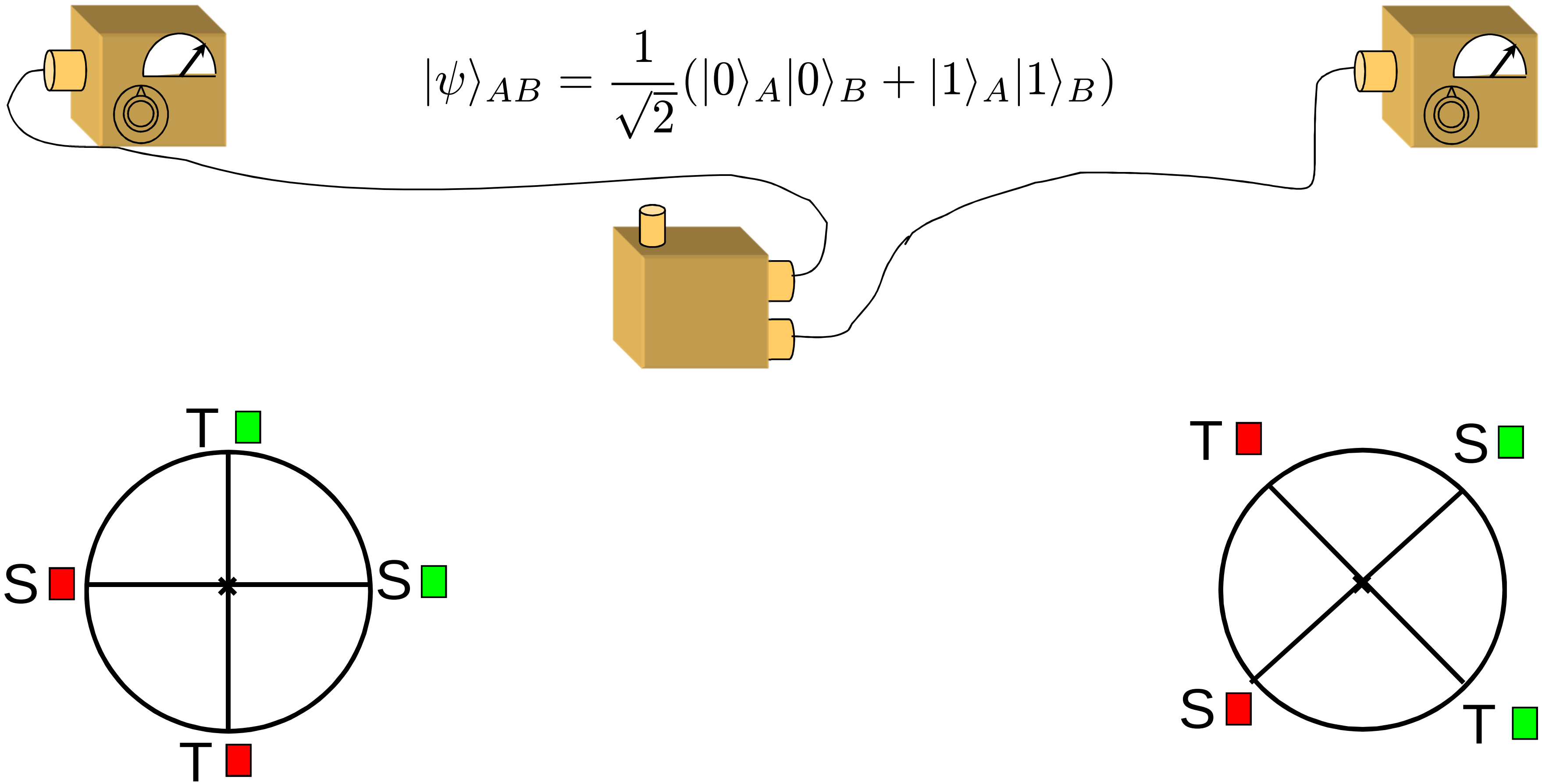
$$\{\xi_0(\lambda_A), \xi_1(\lambda_A)\} \qquad \nu(\lambda_B) \longrightarrow \mu_0(\lambda_B) \text{ or } \mu_1(\lambda_B)$$

where $\frac{1}{2}\mu_0(\lambda_{\mathbf{B}}) + \frac{1}{2}\mu_1(\lambda_{\mathbf{B}}) = \nu(\lambda_{\mathbf{B}})$

$$\{\xi_+(\lambda_A), \xi_-(\lambda_A)\} \qquad \nu(\lambda_B) \longrightarrow \mu_+(\lambda_B) \text{ or } \mu_-(\lambda_B)$$

where $\frac{1}{2}\mu_+(\lambda_{\mathbf{B}}) + \frac{1}{2}\mu_-(\lambda_{\mathbf{B}}) = \nu(\lambda_{\mathbf{B}})$

In this context, locality → preparation noncontextuality



$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$
T
S
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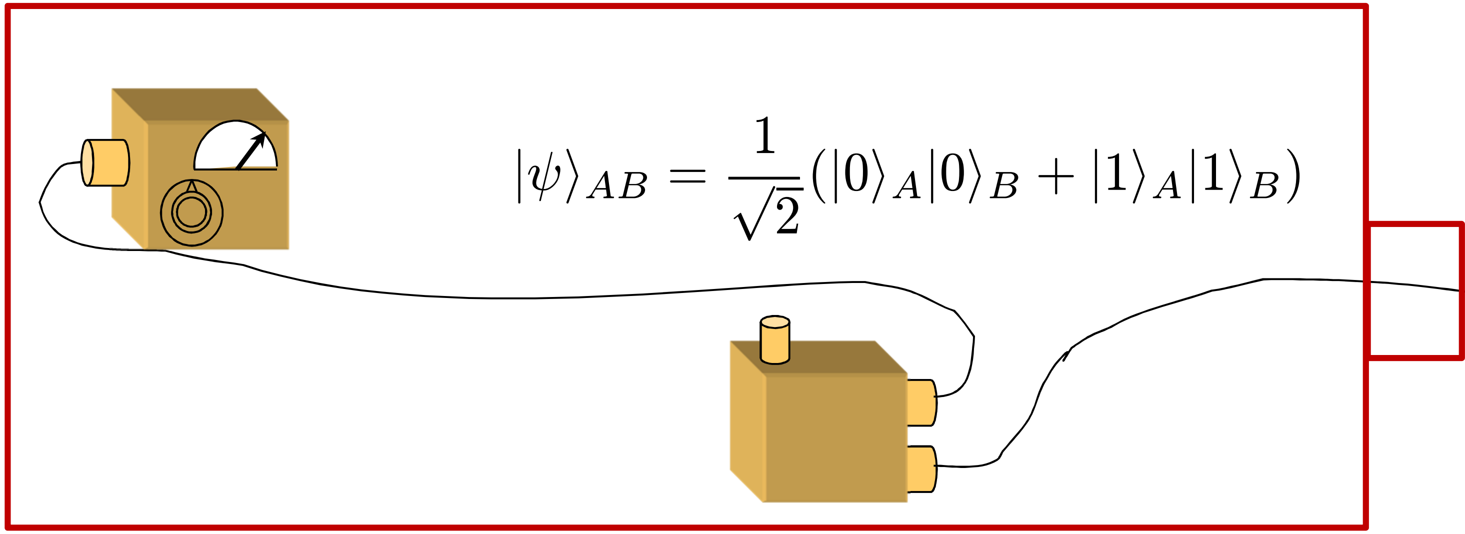

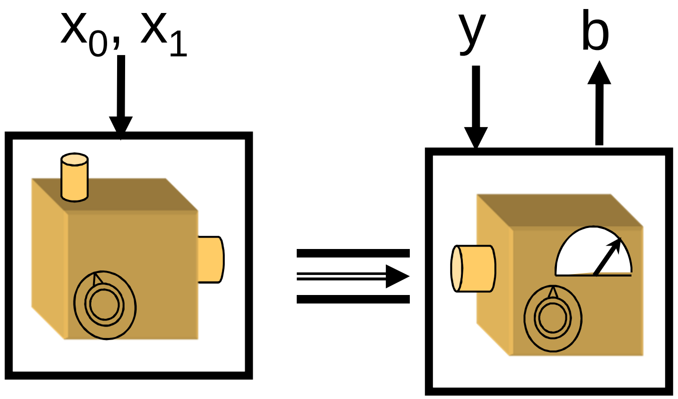

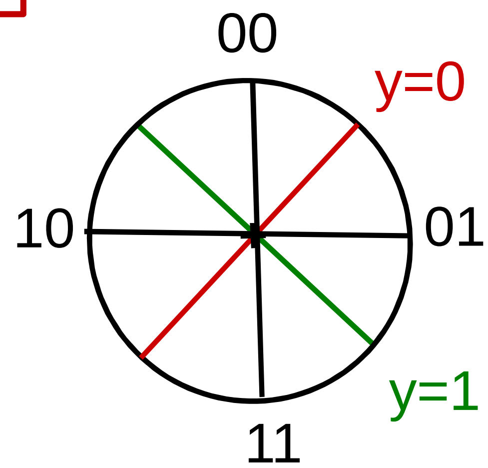

Here, locality → preparation noncontextuality → contradiction

This proof of preparation contextuality → proof of nonlocality

Steering cannot be classical Bayesian updating of hidden variables