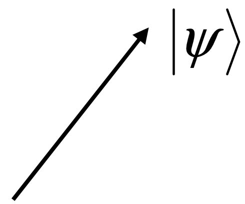
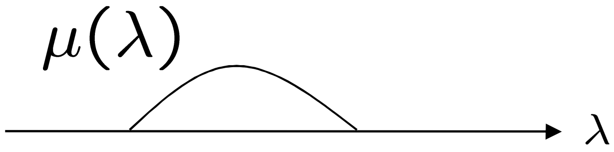
The traditional notion of noncontextuality in quantum theory

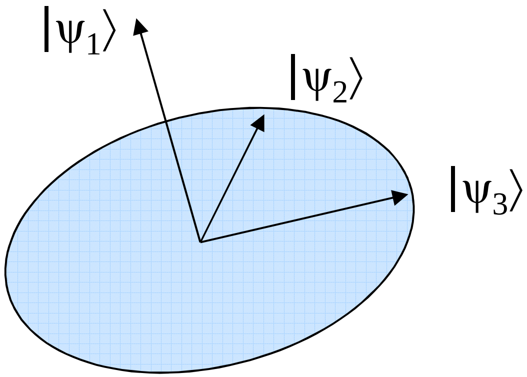
Deterministic hidden variable model for pure states and projective measurements

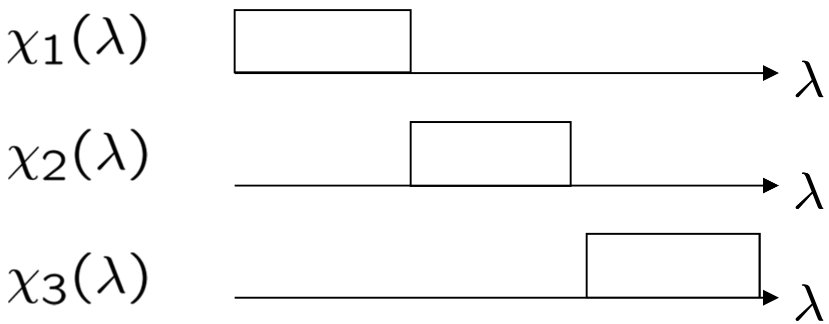
$$\int \mu(\lambda) d\lambda = 1$$





$$\sum_k \chi_k(\lambda) = 1 \text{ for all } \lambda$$



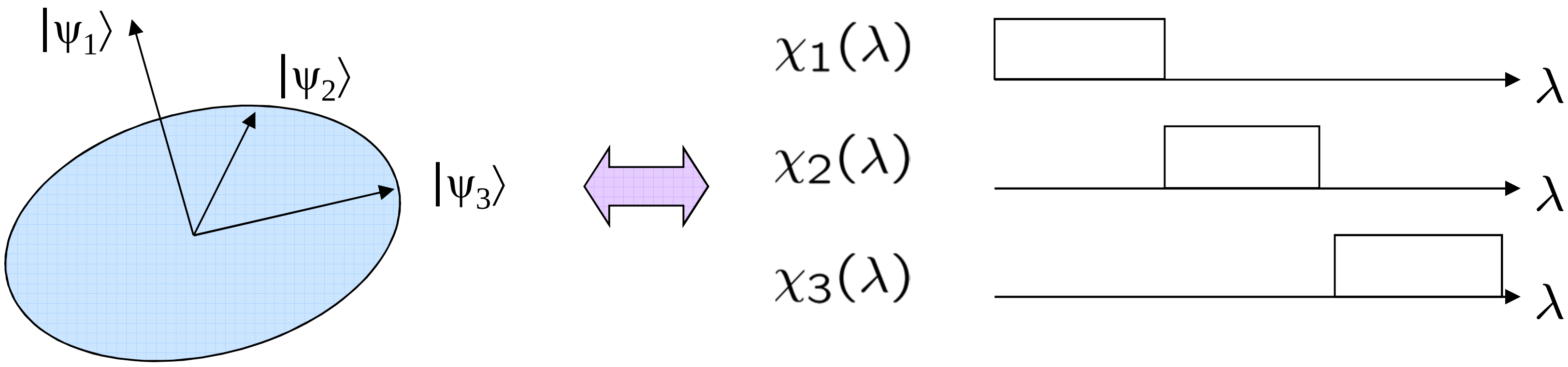


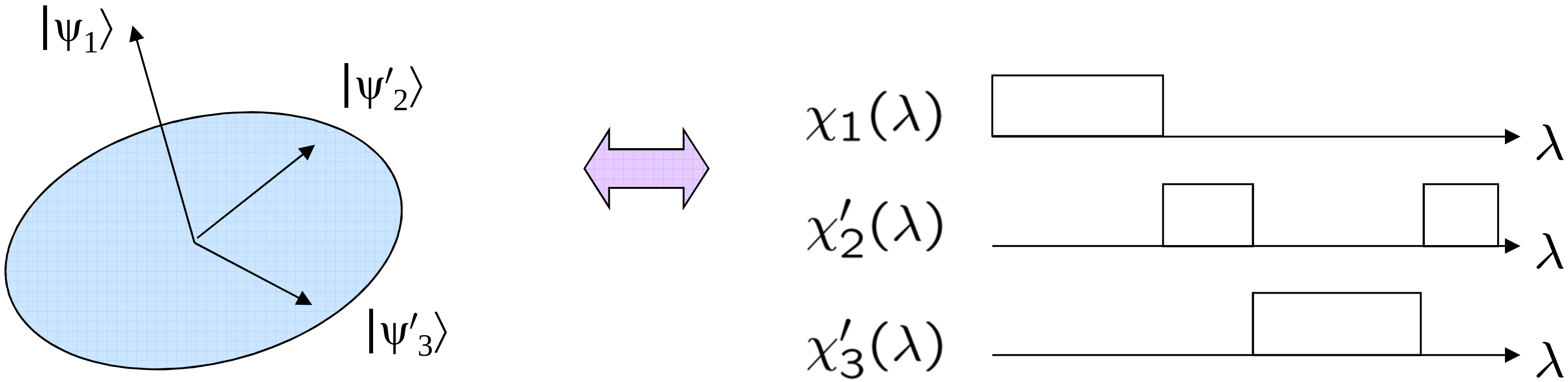
Note: the outcomes are deterministic given λ

$$|\langle\psi|\psi_k\rangle|^2 = \int d\lambda \mu(\lambda) \chi_k(\lambda)$$

Traditional notion of noncontextuality

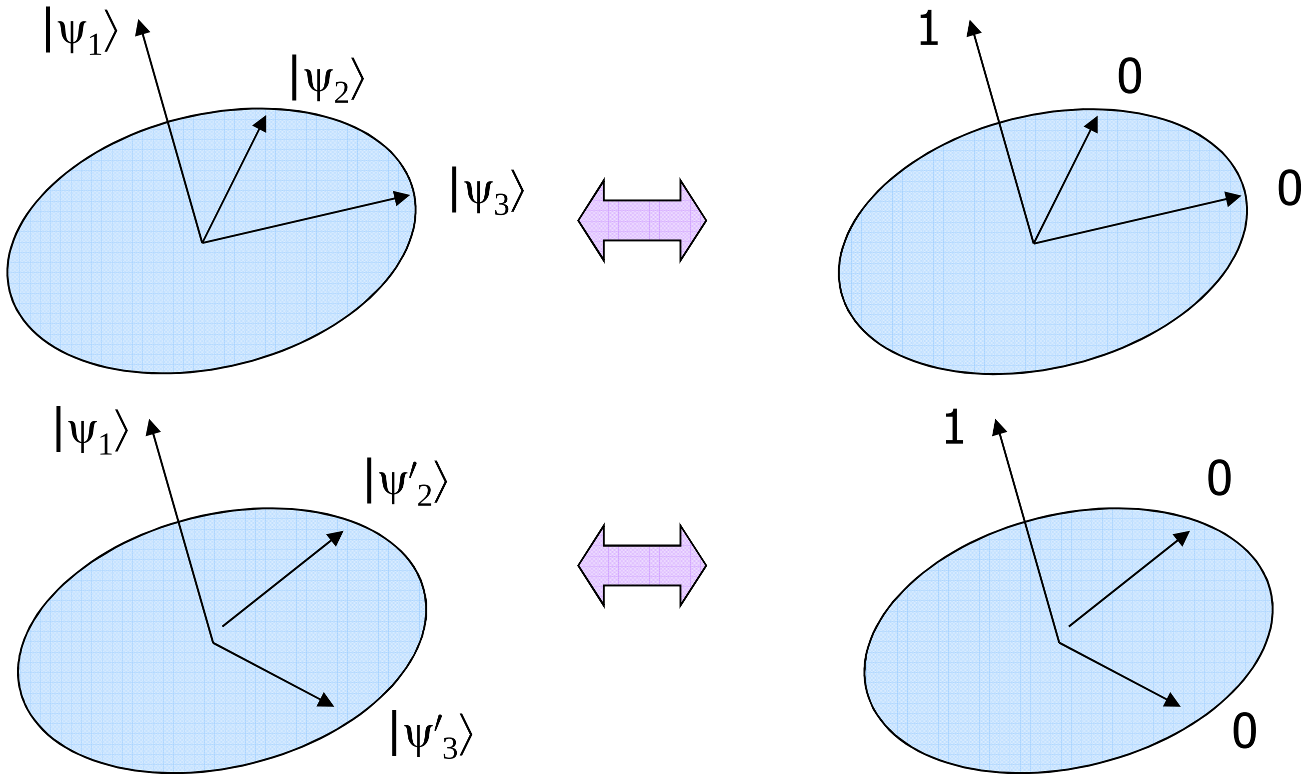
A given vector may appear in many different measurements





The traditional notion of noncontextuality:
Every vector is associated with the same $\chi(\lambda)$
regardless of how it is measured (i.e. the context)

The traditional notion of noncontextuality (take 2):
For every λ, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for λ), and every vector is assigned the same value regardless of which basis it is considered a part (i.e. the context).



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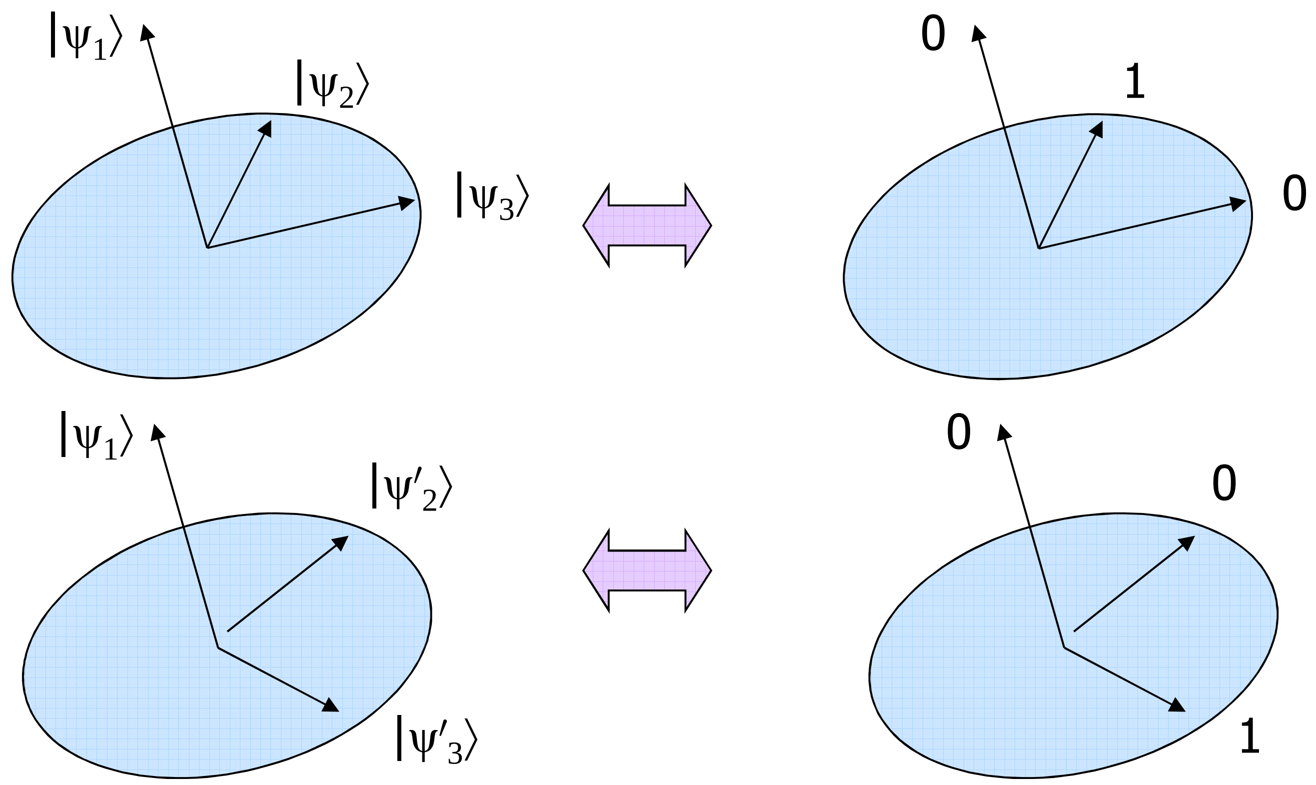
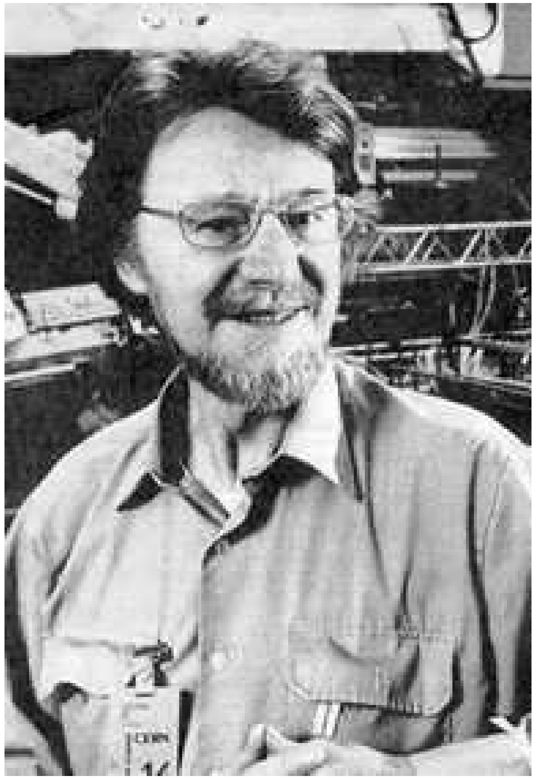




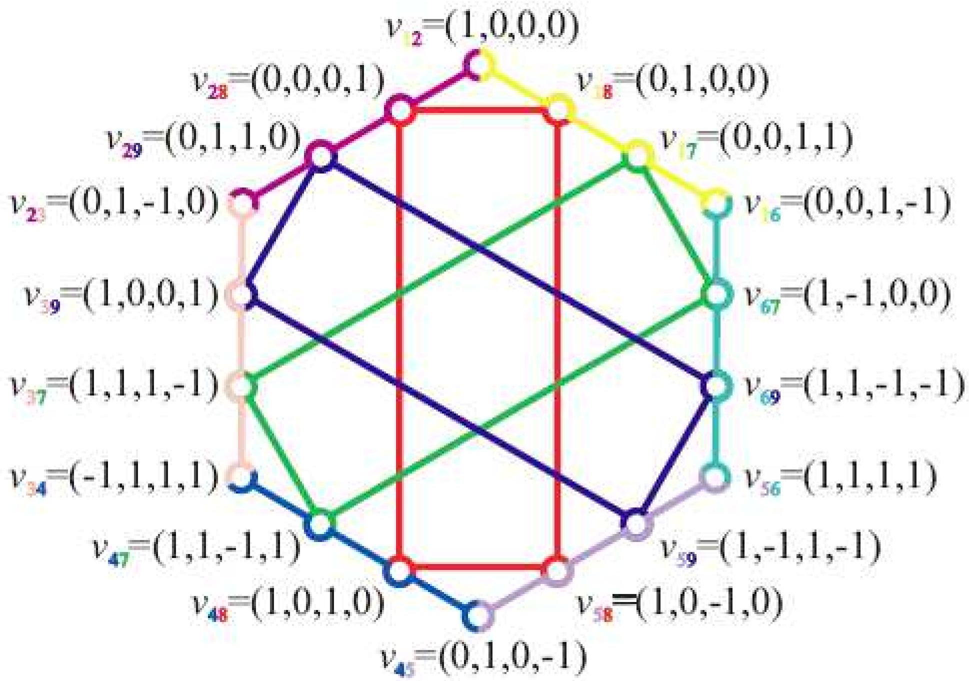
John S. Bell

Ernst Specker (with son) and
Simon Kochen

Bell-Kochen-Specker theorem: A traditional noncontextual hidden variable model of quantum theory for Hilbert spaces of dimension 3 or greater is impossible.

Example: The CEGA 18 ray proof in 4d:

Cabello, Estebaranz, Garcia-Alcaine, Phys. Lett. A 212, 183 (1996)



Example: The CEGA 18 ray proof in 4d:

Cabello, Estebaranz, Garcia-Alcaine, Phys. Lett. A 212, 183 (1996)

If we list all 9 orthogonal quadruples, each ray appears twice in the list

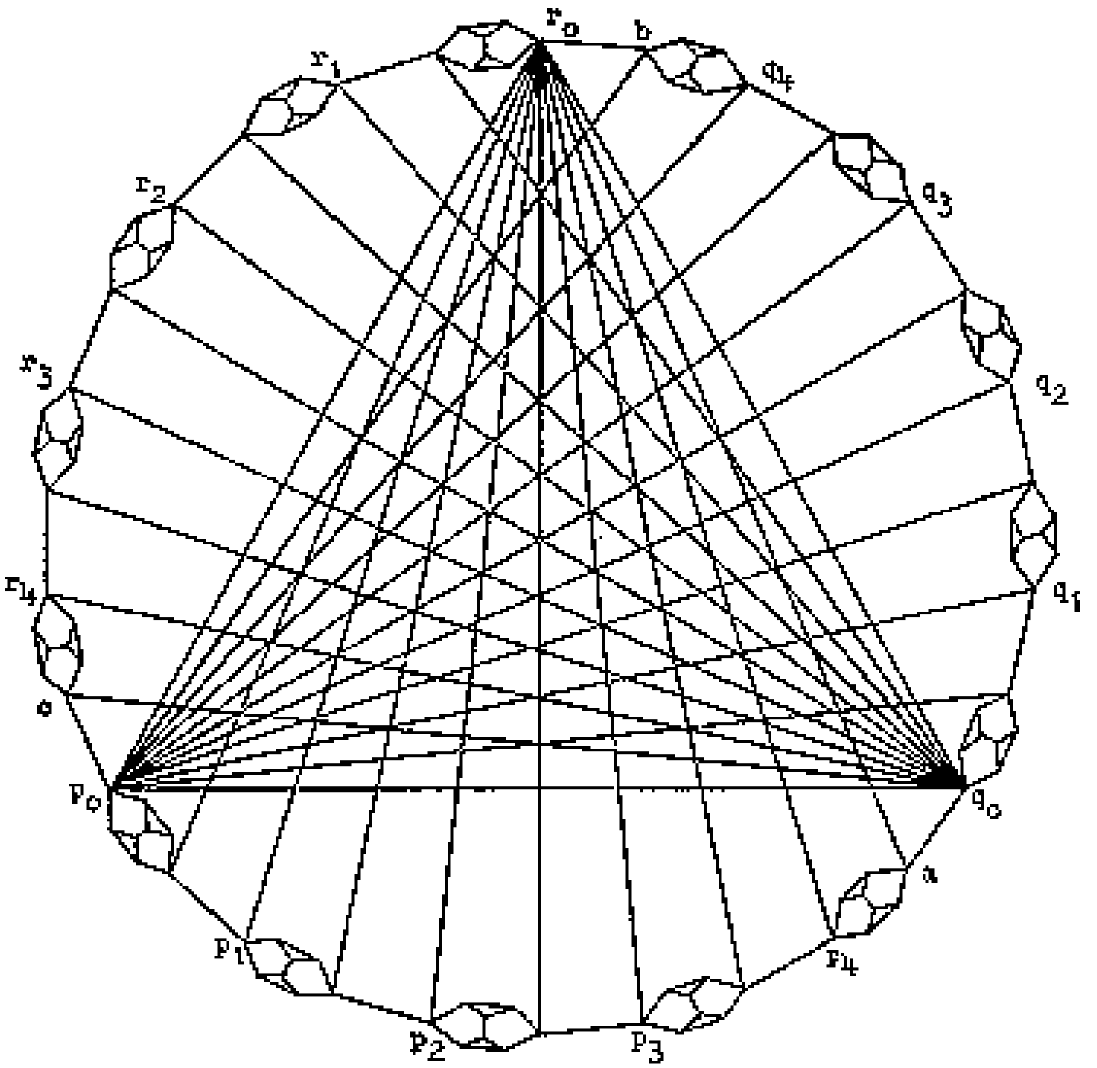
0,0,0,1	0,0,0,1	1,-1,1,-1	1,-1,1,-1	0,0,1,0	1,-1,-1,1	1,1,-1,1	1,1,-1,1	1,1,1,-1
0,0,1,0	0,1,0,0	1,-1,-1,1	1,1,1,1	0,1,0,0	1,1,1,1	1,1,1,-1	-1,1,1,1	-1,1,1,1
1,1,0,0	1,0,1,0	1,1,0,0	1,0,-1,0	1,0,0,1	1,0,0,-1	1,-1,0,0	1,0,1,0	1,0,0,1
1,-1,0,0	1,0,-1,0	0,0,1,1	0,1,0,-1	1,0,0,-1	0,1,-1,0	0,0,1,1	0,1,0,-1	0,1,-1,0

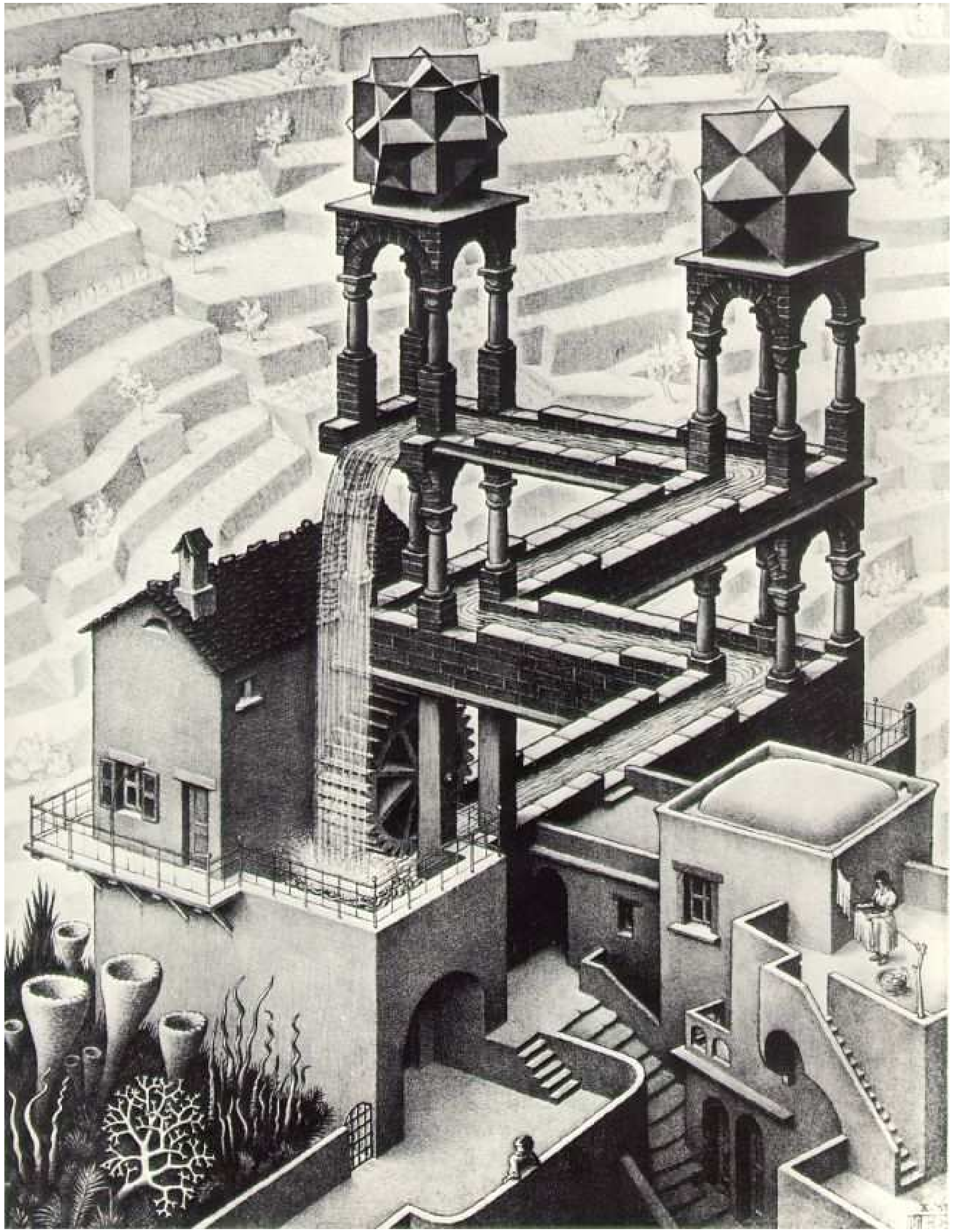
In each of the 9 quadruples, one ray is assigned 1, the other three 0
Therefore, 9 rays must be assigned 1

But each ray appears twice and so there must be an even number of rays assigned 1

CONTRADICTION!

Example: Kochen and Specker’s original 117 ray proof in 3d





The traditional notion of noncontextuality (take 3):
For every λ, every projector Π is assigned a value 0 or 1 regardless of which basis it is a coarse-graining of (i.e. the context)

$$v(\Pi) = 0 \text{ or } 1 \quad \text{for all } \Pi$$

Coarse-graining of a measurement implies a coarse-graining of the value (because it is just post-processing)

$$v(\sum_k \Pi_k) = \sum_k v(\Pi_k)$$

Every measurement has *some* outcome

$$v(I) = 1$$

The traditional notion of noncontextuality (take 4):

For Hermitian operators A, B, C satisfying

$$[A,B]=0 \qquad [A,C]=0 \qquad [B,C]\neq 0$$

the value assigned to A should be independent of whether it is measured together with B or together with C (i.e. the context)

Measure A = measure projectors onto eigenspaces of A, $\{\Pi_a\}$

$$A=\sum_a a\,\Pi_a \quad \rightarrow \quad v(A)=\sum_a a\, v(\Pi_a)$$

Measure A in context of B
= measure projectors onto joint eigenspaces of A and B, $\{\Pi_{ab}\}$
then coarse-grain over B outcome $\Pi_a=\sum_b \Pi_{ab}$

Measure A in context of C
= measure projectors onto joint eigenspaces of A and C, $\{\Pi_{ac}\}$
Then coarse-grain over C outcome $\Pi_a=\sum_c \Pi_{ac}$

$v(\Pi_a)$ is independent of context ➔ $v(A)$ is independent of context

Functional relationships among commuting Hermitian operators must be respected by their values

$$\text{If} \quad f(L, M, N, ...) = 0$$

$$\text{then} \quad f(v(L), v(M), v(N), ...) = 0$$

Proof: the possible sets of eigenvalues one can simultaneously assign to L, M, N,… are specified by their joint eigenstates. By acting the first equation on each of the joint eigenstates, we get the second.

Example: Mermin's magic square proof in 4d

X_1	X_2	X_1X_2	I
Y_2	Y_1	Y_1Y_2	I
X_1Y_2	Y_1X_2	Z_1Z_2	I
I	I	$-I$	

$$X_1\ X_2\ (X_1X_2) = I$$
$$Y_1\ Y_2\ (Y_1Y_2) = I$$
$$(X_1Y_2)\ (Y_1X_2)\ (Z_1Z_2) = I$$
$$X_1\ Y_2\ (X_1Y_2) = I$$
$$Y_1\ X_2\ (Y_1X_2) = I$$
$$(X_1X_2)\ (Y_1Y_2)\ (Z_1Z_2) = -I$$

$$v(X_1)\ v(X_2)\ v(X_1X_2) = 1$$
$$v(Y_1)\ v(Y_2)\ v(Y_1Y_2) = 1$$
$$v(X_1Y_2)\ v(Y_1X_2)\ v(Z_1Z_2) = 1$$
$$v(X_1)\ v(Y_2)\ v(X_1Y_2) = 1$$
$$v(Y_1)\ v(X_2)\ v(Y_1X_2) = 1$$
$$v(X_1X_2)\ v(Y_1Y_2)\ v(Z_1Z_2) = -1$$

Product of LHSs = +1

Product of RHSs = -1

CONTRADICTION

Problems with the traditional definition of noncontextuality:

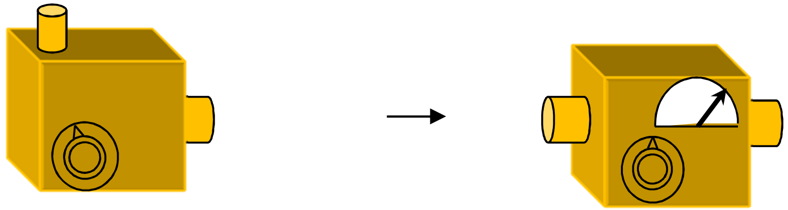
- applies only to *projective* measurements
- applies only to *deterministic* hidden variable models
- applies only to models of *quantum theory*

An operational notion of noncontextuality would determine

- whether any given operational theory admits of a noncontextual model
- whether any given experimental data can be explained by a noncontextual model

The traditional notion of noncontextuality extended to any operational theory

Operational theories



Preparation
P

Measurement
M

These are defined as lists of instructions

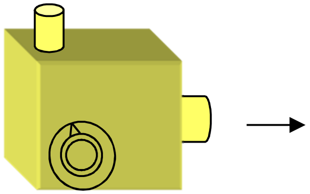
An operational theory specifies

$p(k|\mathsf{P},\mathsf{M}) \equiv$ The probability of outcome k of M given P

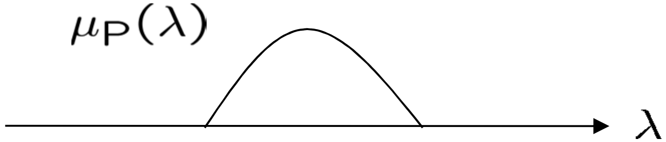
A deterministic hidden variable model of an operational theory

Specifies an ontic state space Λ

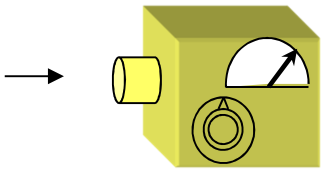
Preparation
P



$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$

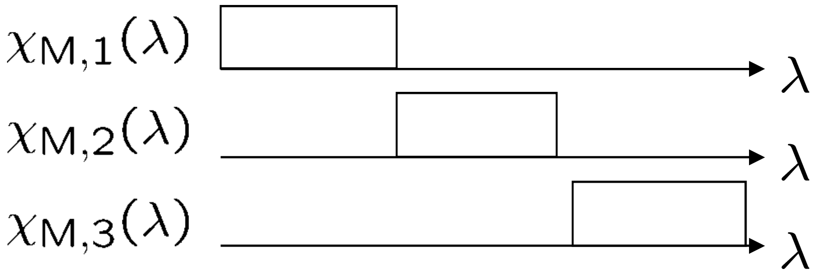


Measurement
M

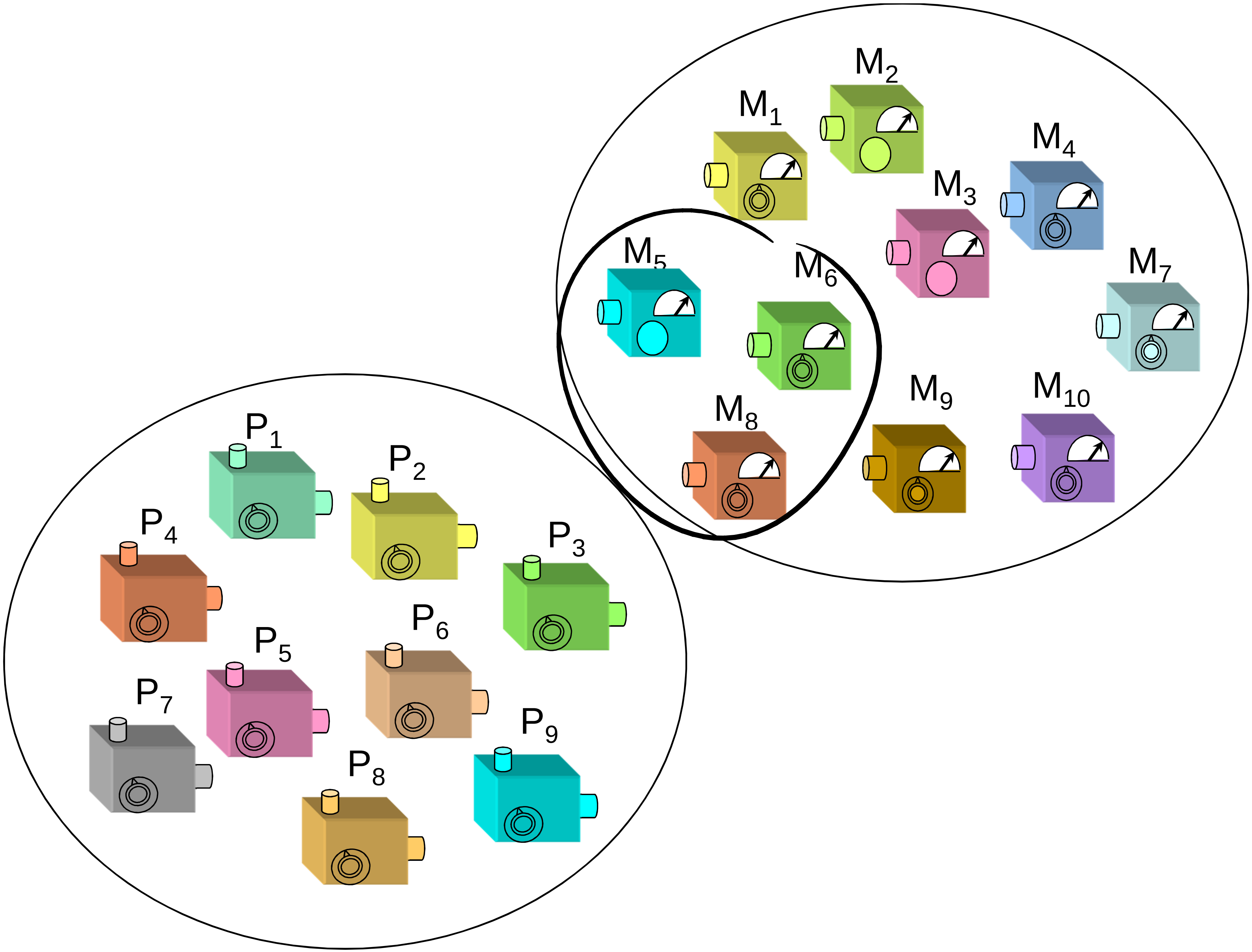


$\chi_{\mathsf{M},k} \in \{0, 1\}$

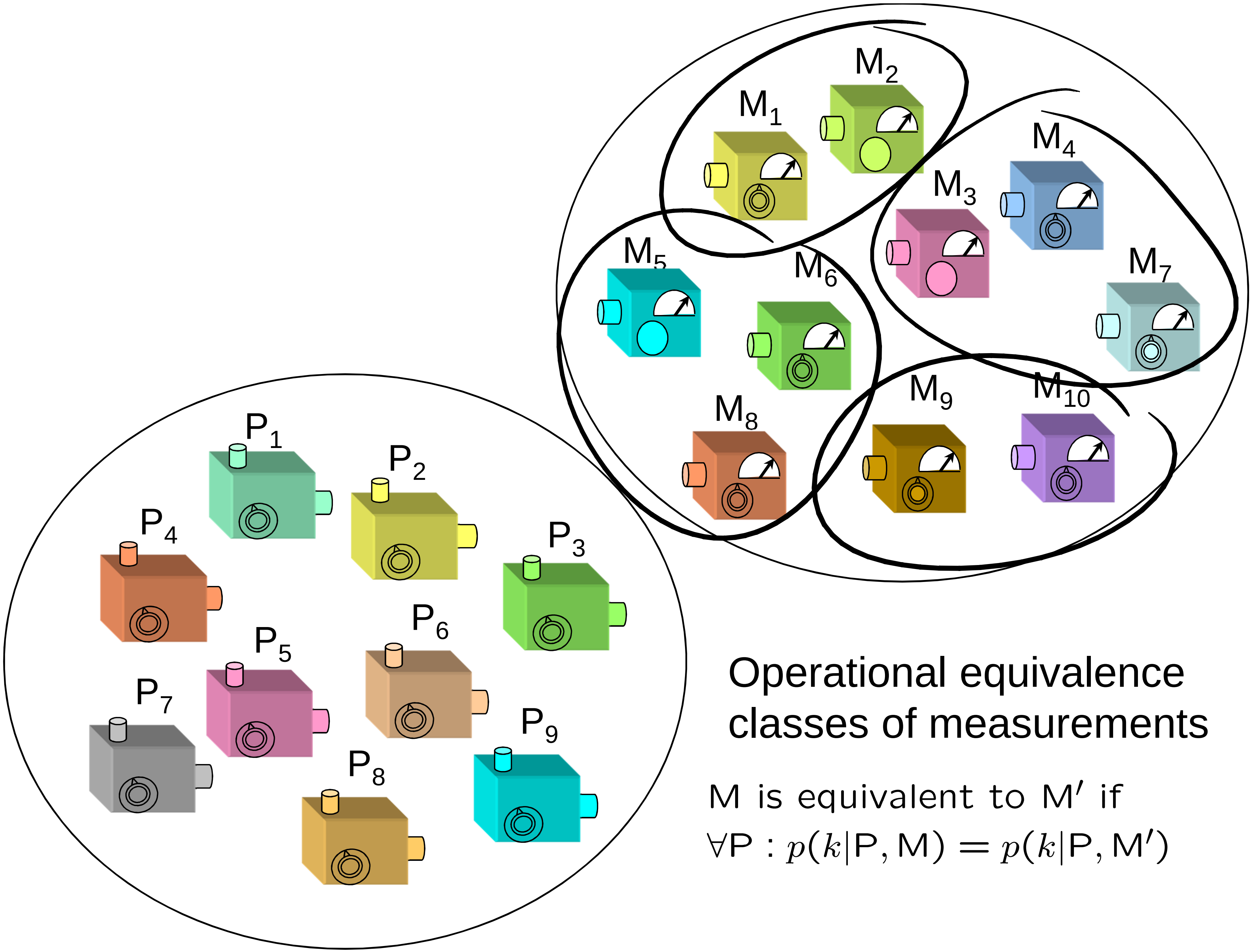
$\sum_k \chi_{\mathsf{M},k}(\lambda) = 1$ for all λ



$$p(k|\mathsf{P},\mathsf{M}) = \int d\lambda \; \chi_{\mathsf{M},k}(\lambda) \; \mu_{\mathsf{P}}(\lambda)$$



M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9

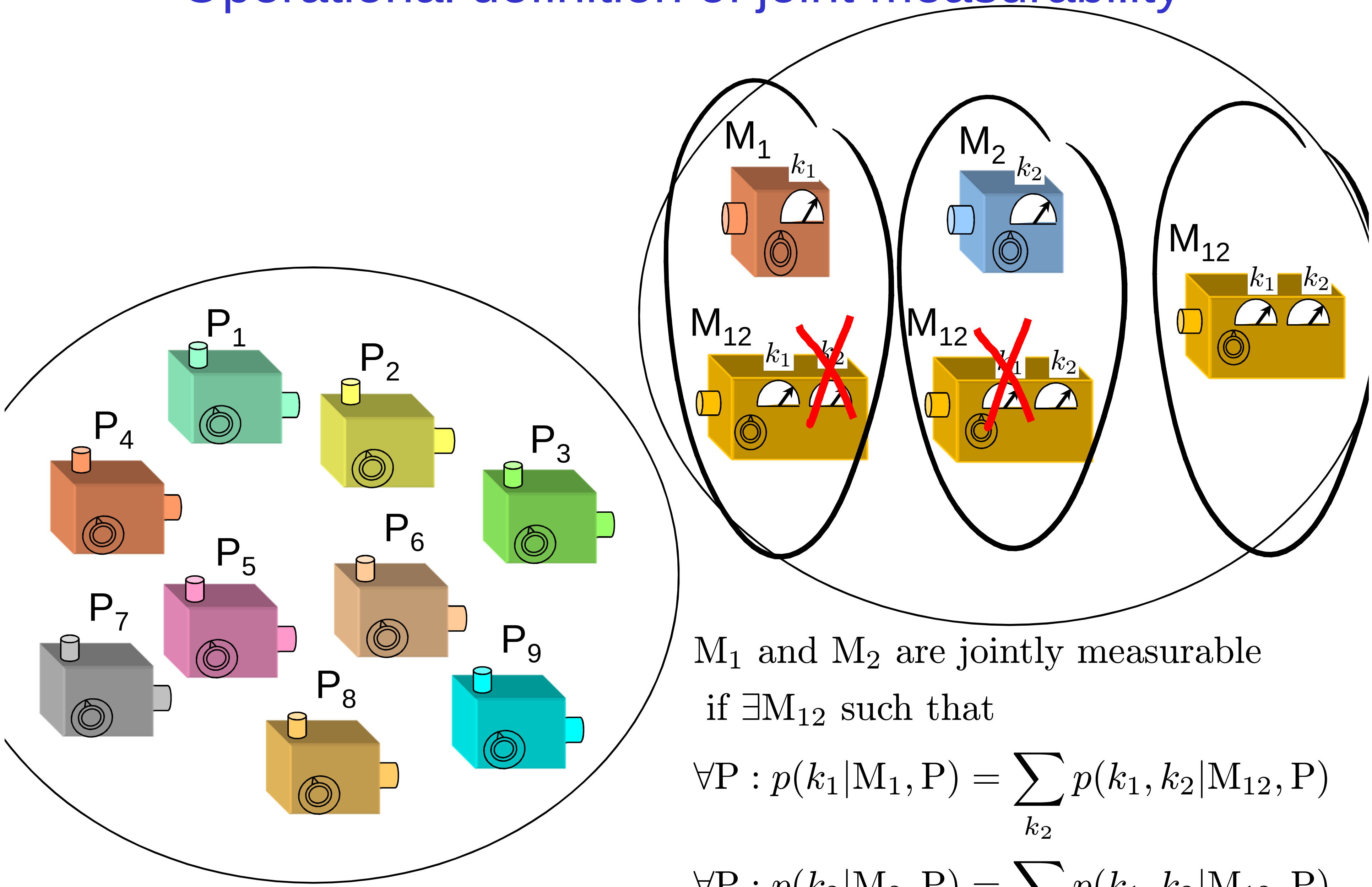


Operational equivalence classes of measurements

M is equivalent to M′ if

$\forall \mathrm{P} : p(k|\mathrm{P},\mathrm{M}) = p(k|\mathrm{P},\mathrm{M}')$

Operational definition of joint measurability

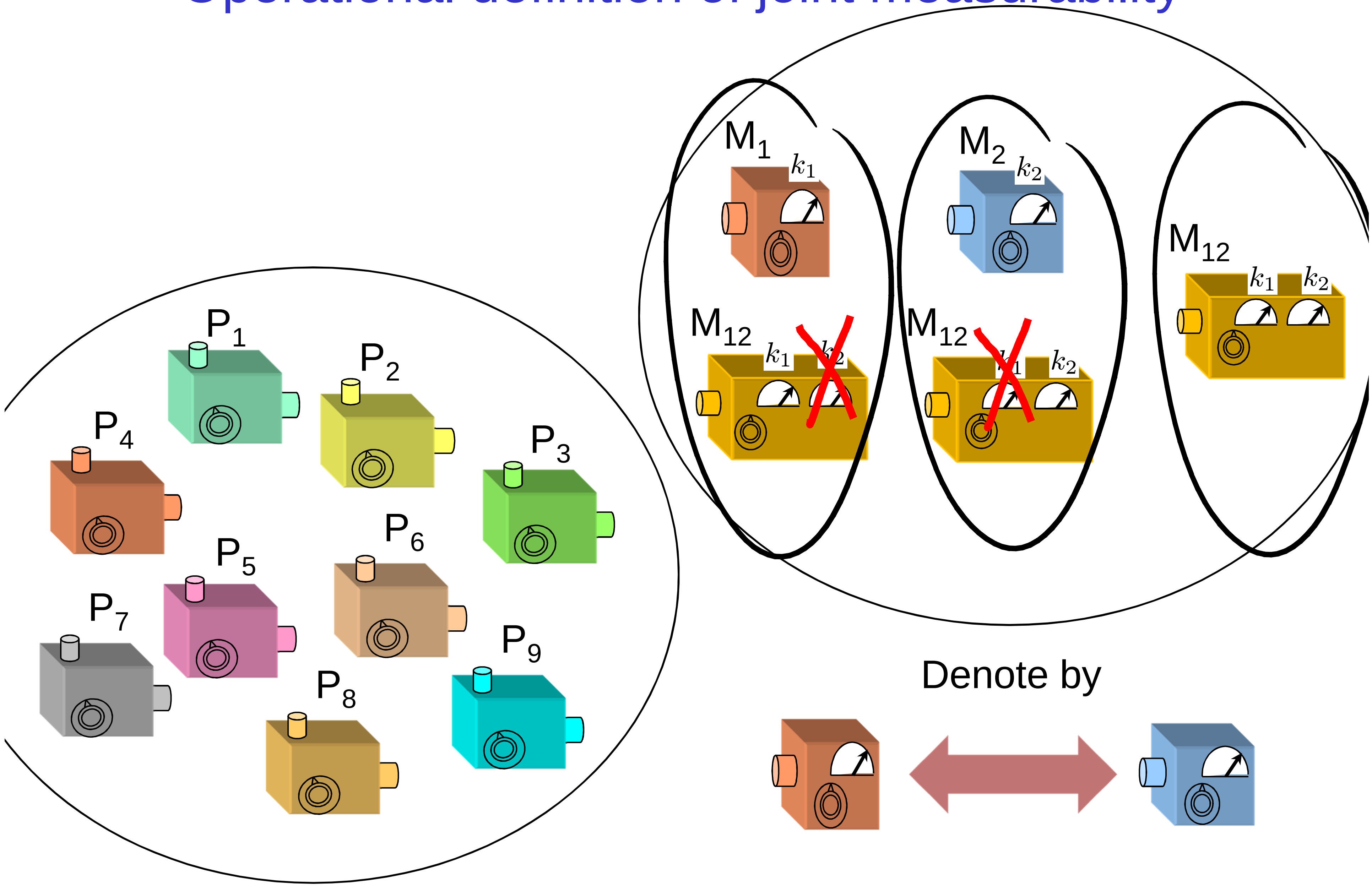


M_1 and M_2 are jointly measurable
if $\exists \mathrm{M}_{12}$ such that

$$\forall \mathrm{P} : p(k_1|\mathrm{M}_1, \mathrm{P}) = \sum_{k_2} p(k_1, k_2|\mathrm{M}_{12}, \mathrm{P})$$

$$\forall \mathrm{P} : p(k_2|\mathrm{M}_2, \mathrm{P}) = \sum_{k_1} p(k_1, k_2|\mathrm{M}_{12}, \mathrm{P})$$

Operational definition of joint measurability



Definition of a traditionally noncontextual hidden variable model for an operational theory

One for which:

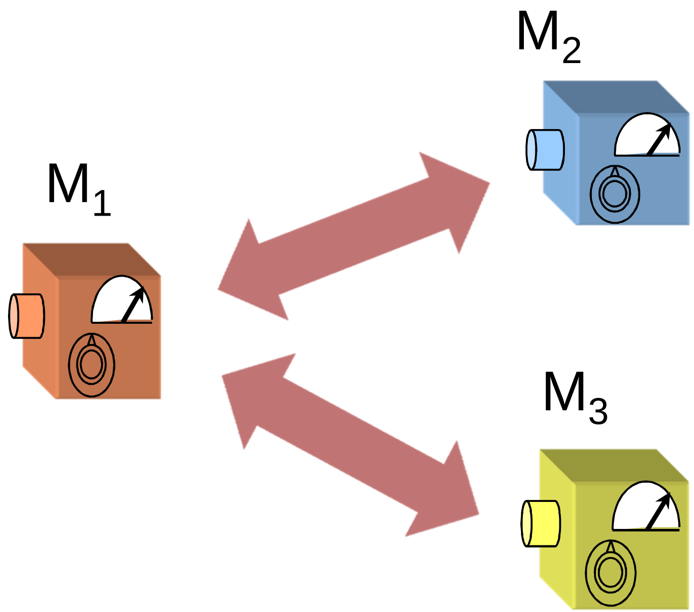
Outcomes are fixed deterministically by the ontic state λ

Outcomes are independent of the context of the measurement

Example:

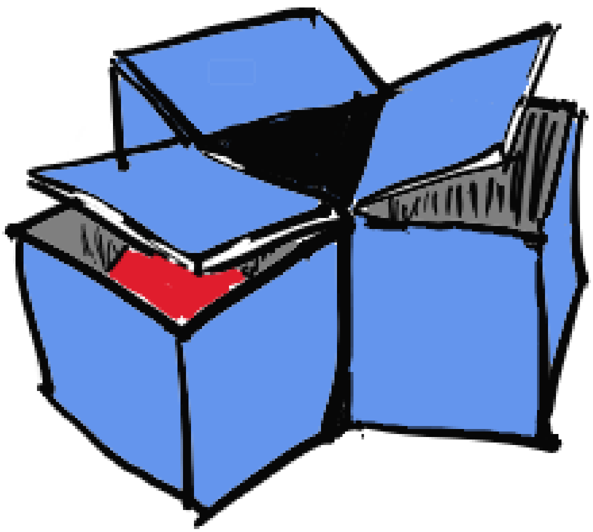
M_1 and M_2 jointly measurable

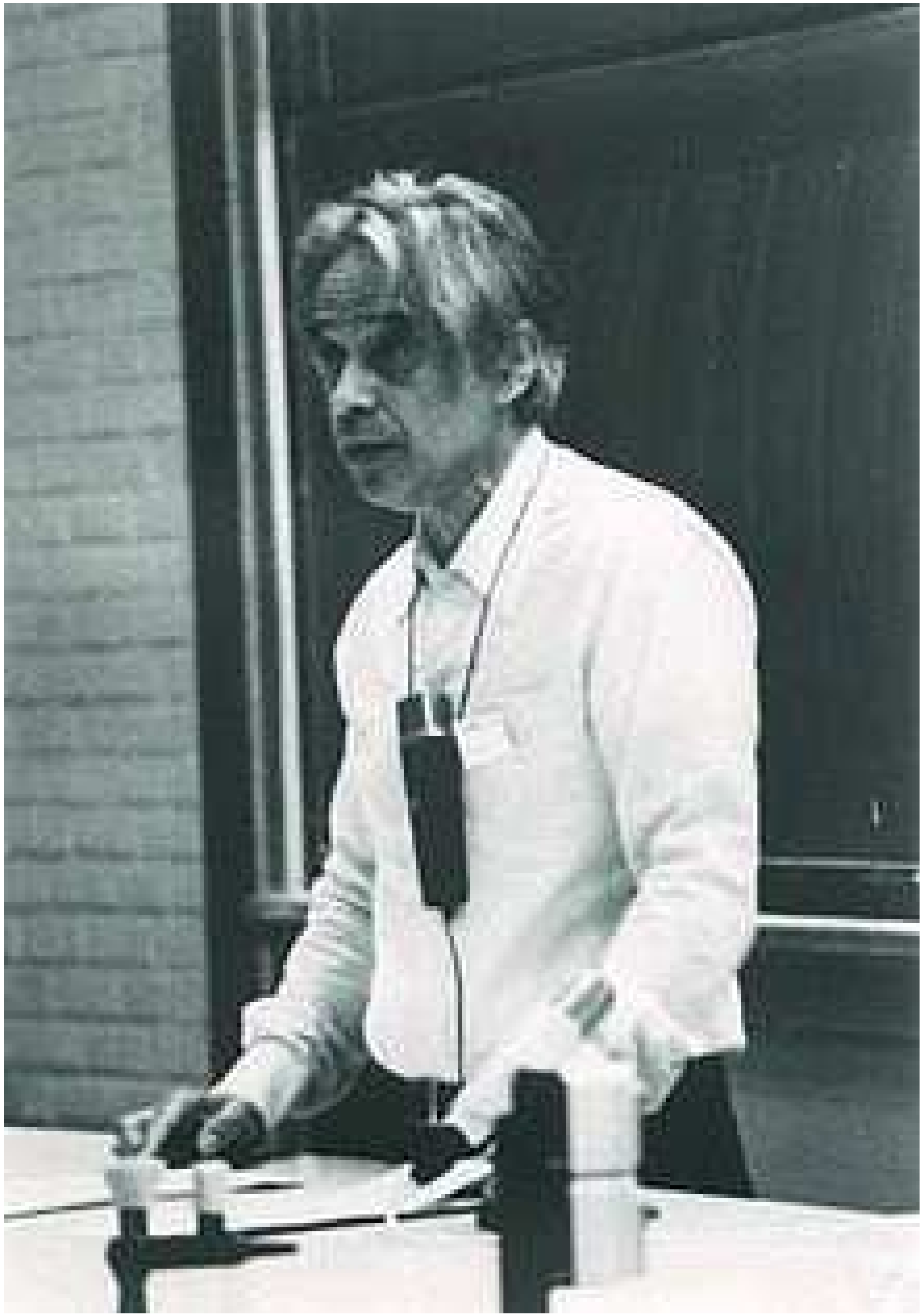
M_1 and M_3 jointly measurable



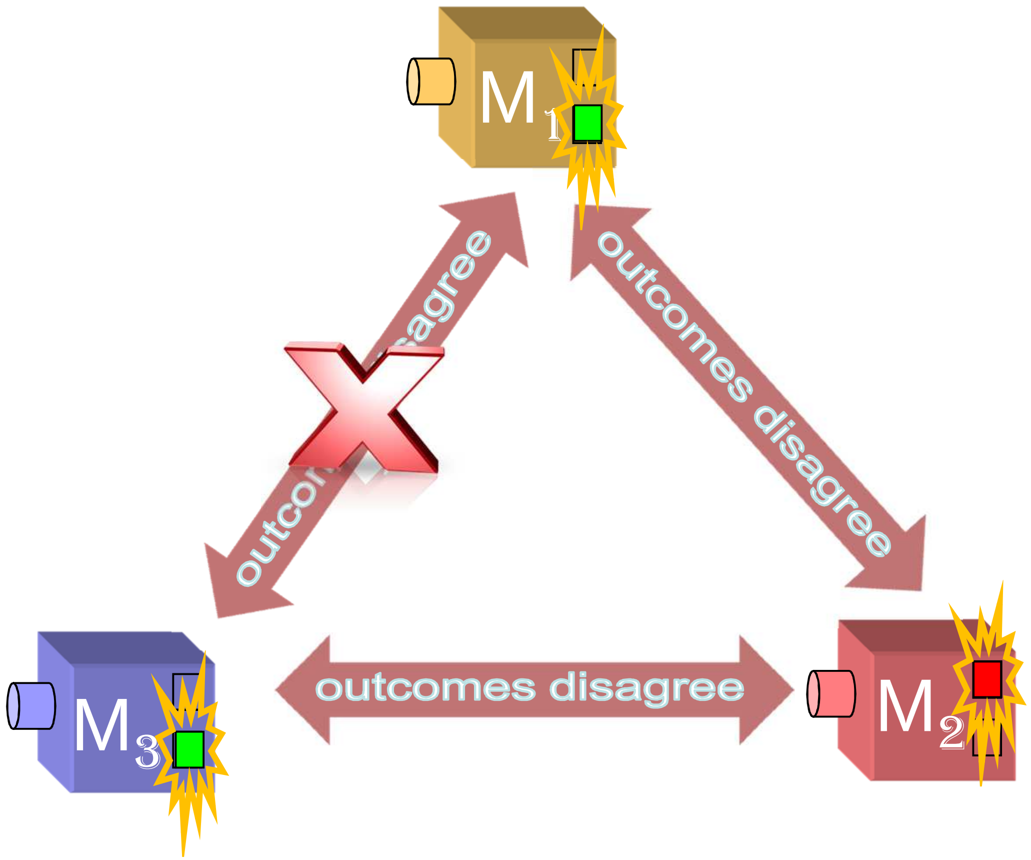
Outcome assigned to M_1 by λ is independent of context

Ernst Specker, “The logic of propositions which are not simultaneously decidable”, Dialectica 14, 239 (1960).





Specker's example



If the outcomes are fixed deterministically by the ontic state and are independent of the context in which the measurement is performed, then

$$p(\text{success}) \leq \frac{2}{3}$$

Frustrated Networks

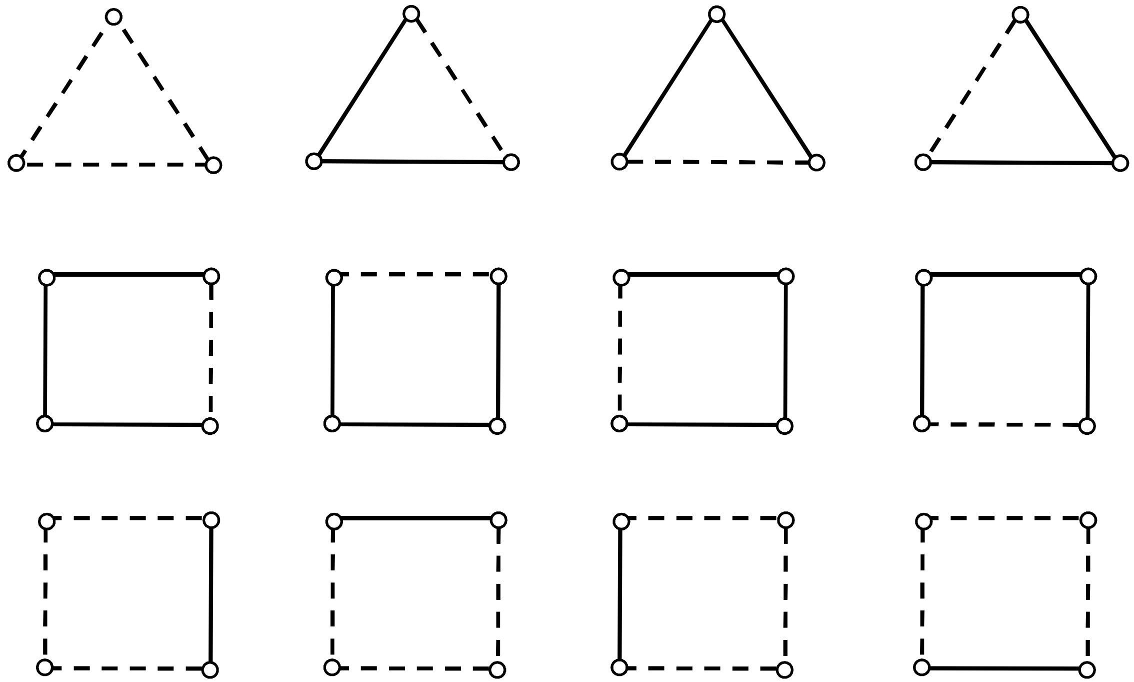
Nodes are binary variables

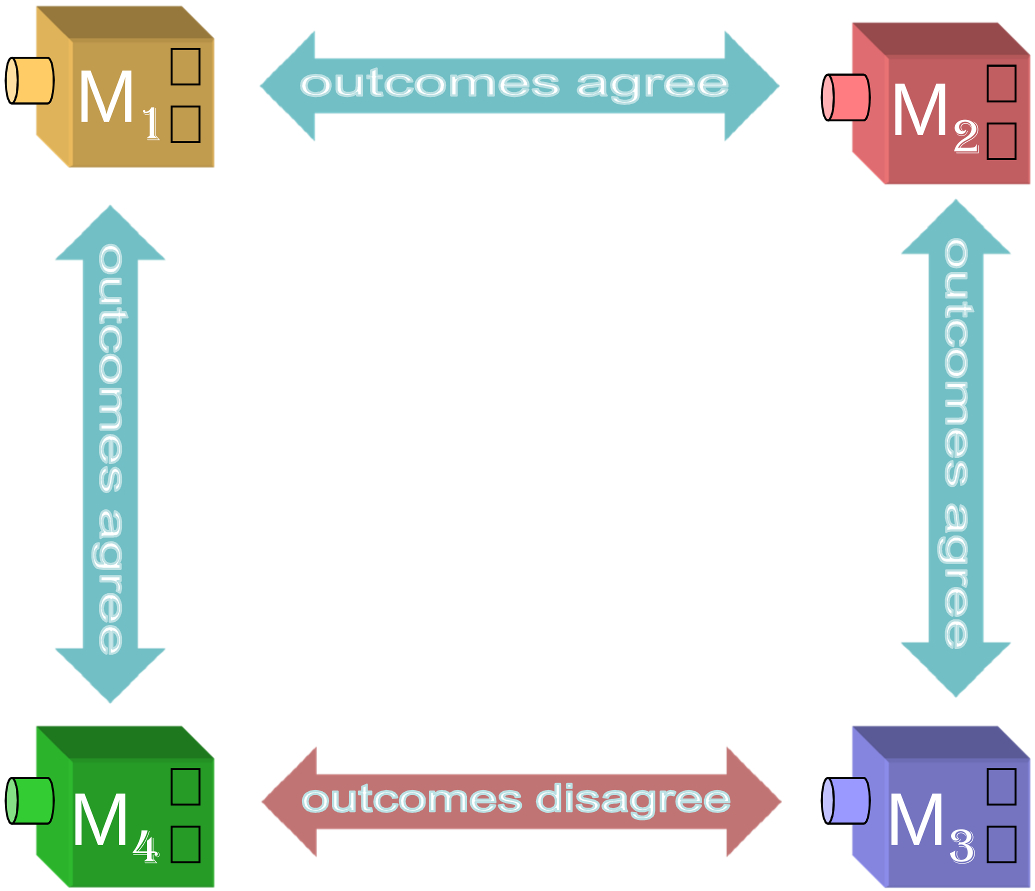
Edges imply joint measurability

 Outcomes agree

 Outcomes disagree

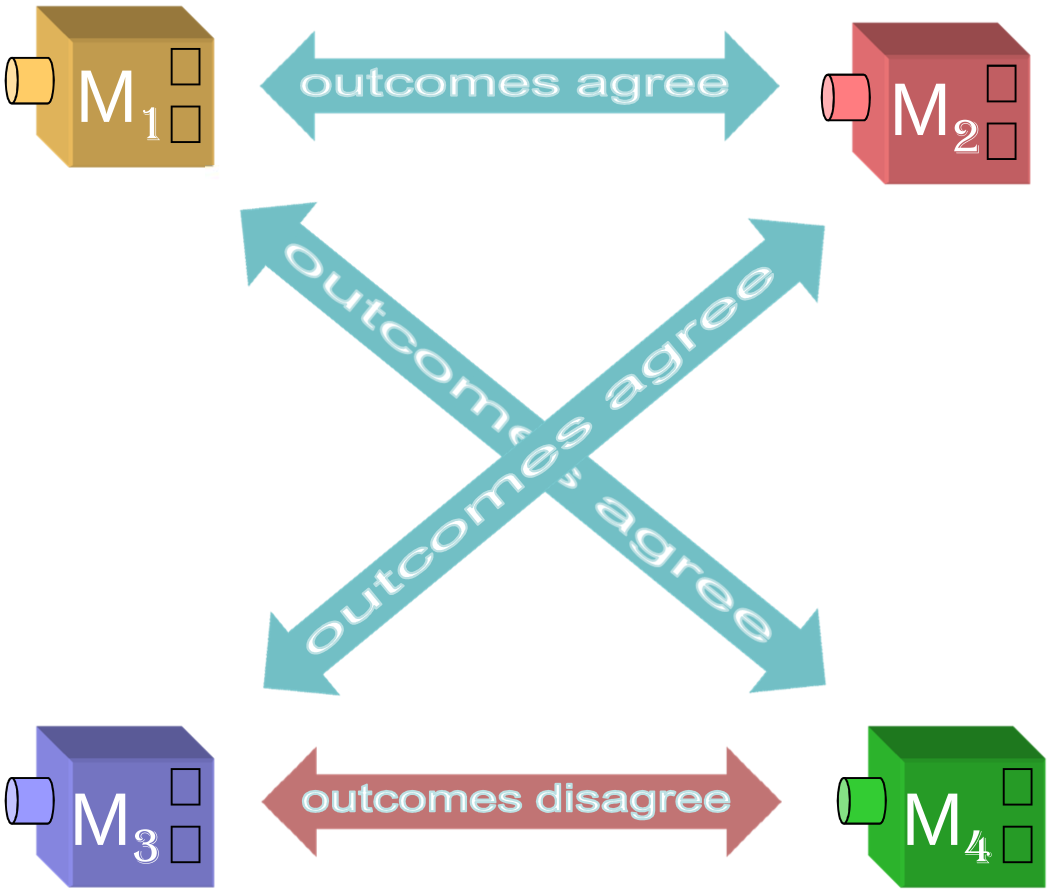
Frustration = no valuation satisfying all correlations

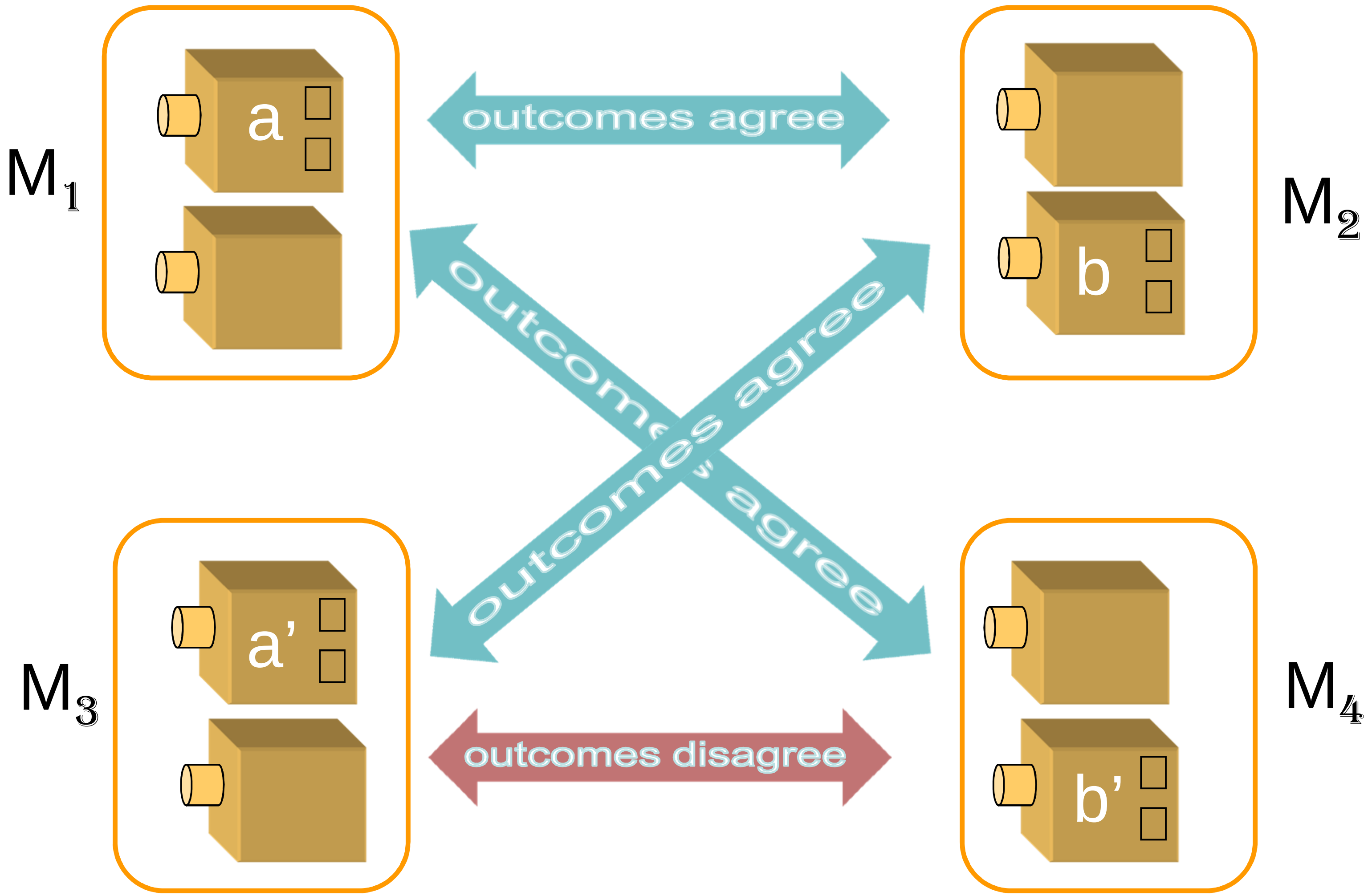




If the outcomes are fixed deterministically by the ontic state and are independent of the context in which the measurement is performed, then

$$p(\text{success}) \leq \frac{3}{4}$$

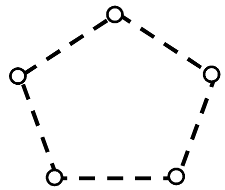

M1
M2
M3
M4
outcomes agree
outcomes agree
outcomes agree
outcomes disagree



Locality + Determinism

→ independence of outcomes on remote contexts

Klyachko's example



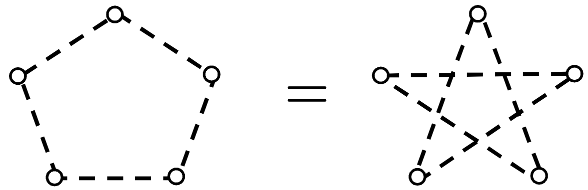
$p(\text{success}) \leq \frac{4}{5}$

5 projective mmts:

$$\{|l_1\rangle\langle l_1| \ , \ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2| \ , \ I - |l_2\rangle\langle l_2|\}$$
$$\{|l_3\rangle\langle l_3| \ , \ I - |l_3\rangle\langle l_3|\}$$
$$\{|l_4\rangle\langle l_4| \ , \ I - |l_4\rangle\langle l_4|\}$$
$$\{|l_5\rangle\langle l_5| \ , \ I - |l_5\rangle\langle l_5|\}$$

where $\langle l_i | l_{i\oplus 1}\rangle = 0 \quad i \in \{1, \ldots, 5\}$

Klyachko's example



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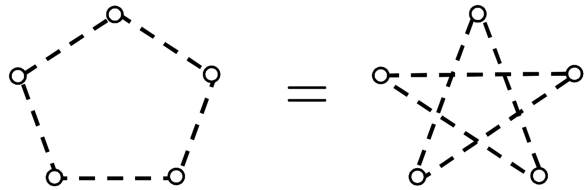

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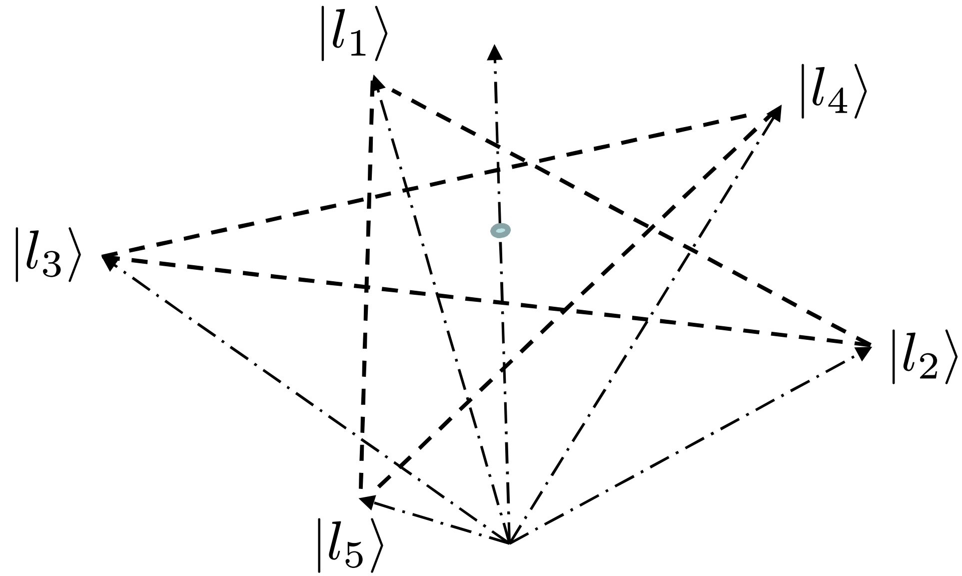

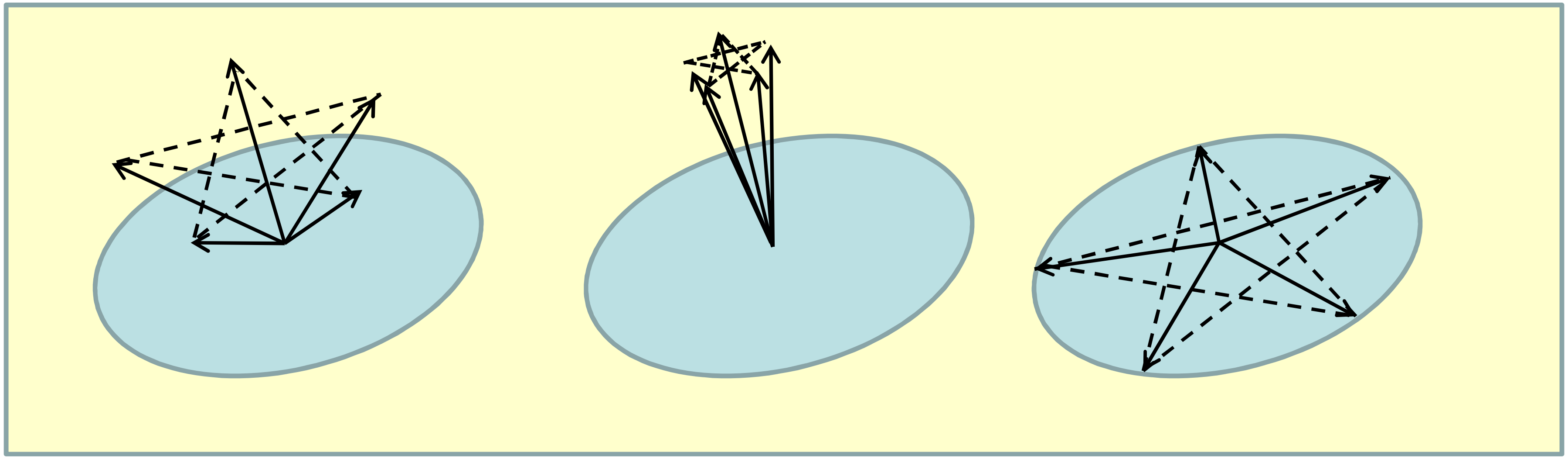

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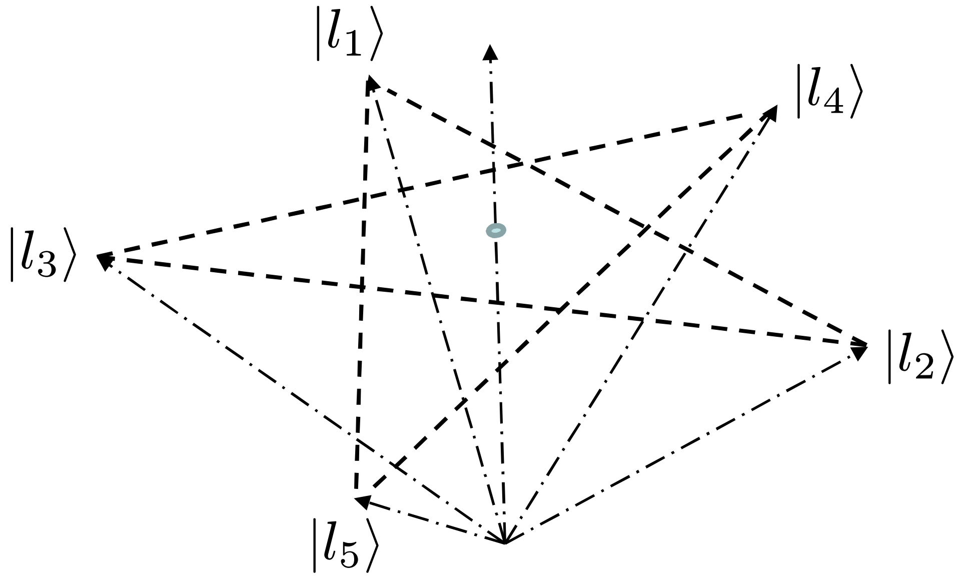
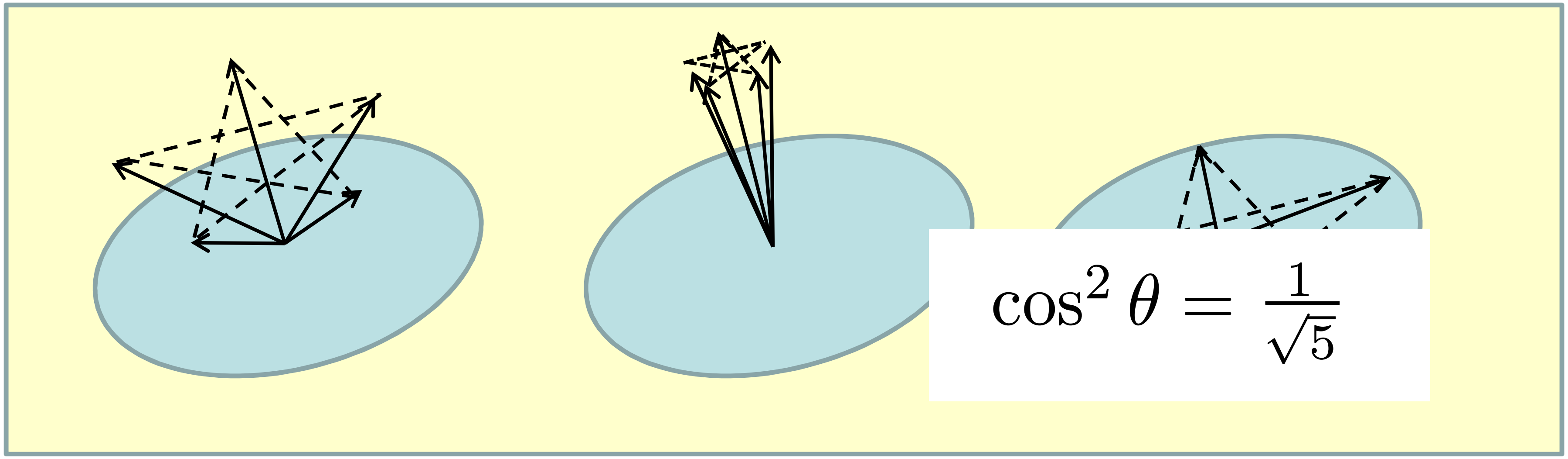
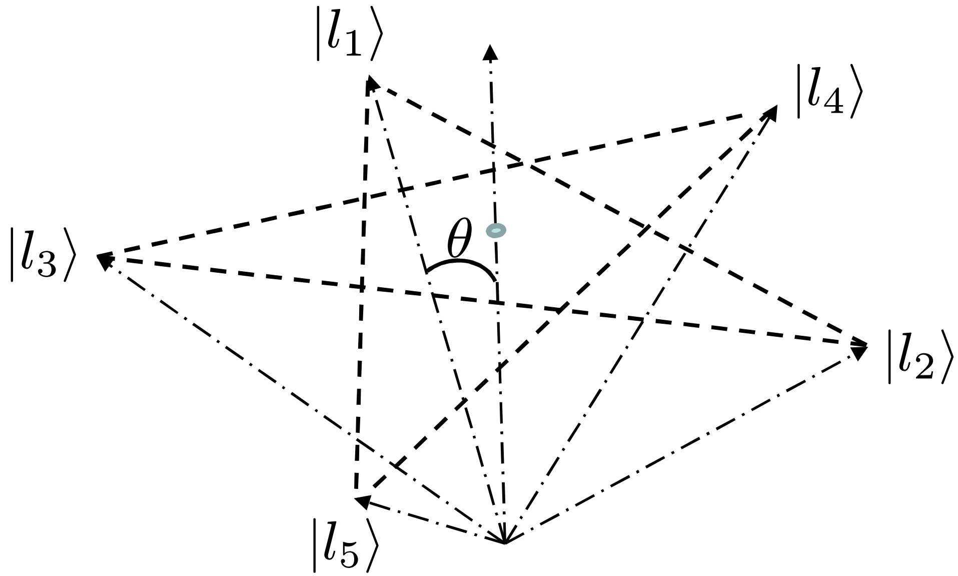


where $\langle l_i | l_{i\oplus 1}\rangle = 0 \quad i \in \{1, \ldots, 5\}$



$$\cos^2\theta = \frac{1}{\sqrt{5}}$$

5 projective mmts:



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Klyachko's example

$$\cos^2\theta = \frac{1}{\sqrt{5}}$$

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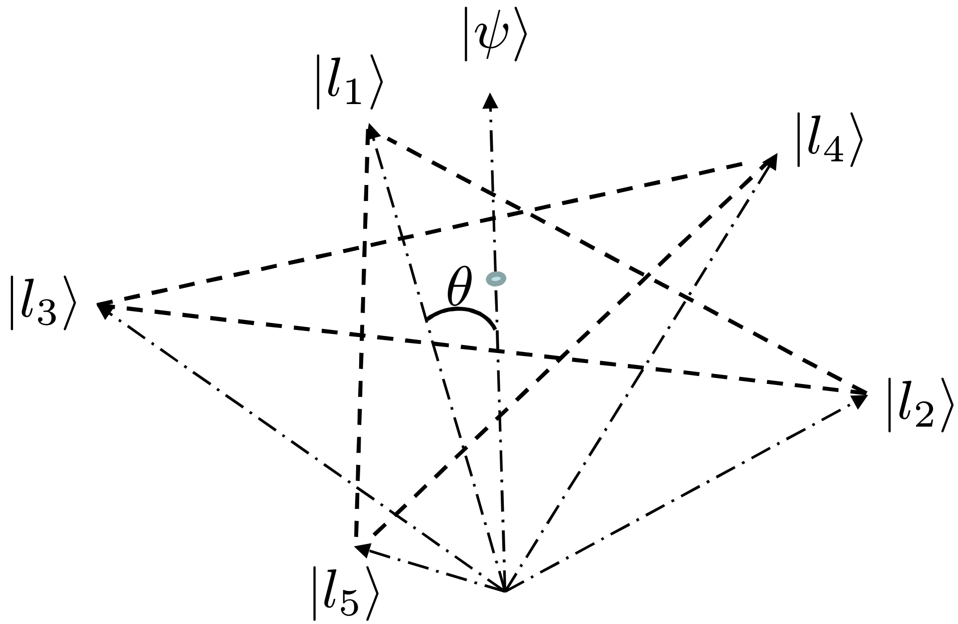


where $\langle l_i | l_{i\oplus 1}\rangle = 0 \quad i \in \{1, \ldots, 5\}$

Preparation: the ψ that lies on the symmetry axis

Klyachko's example

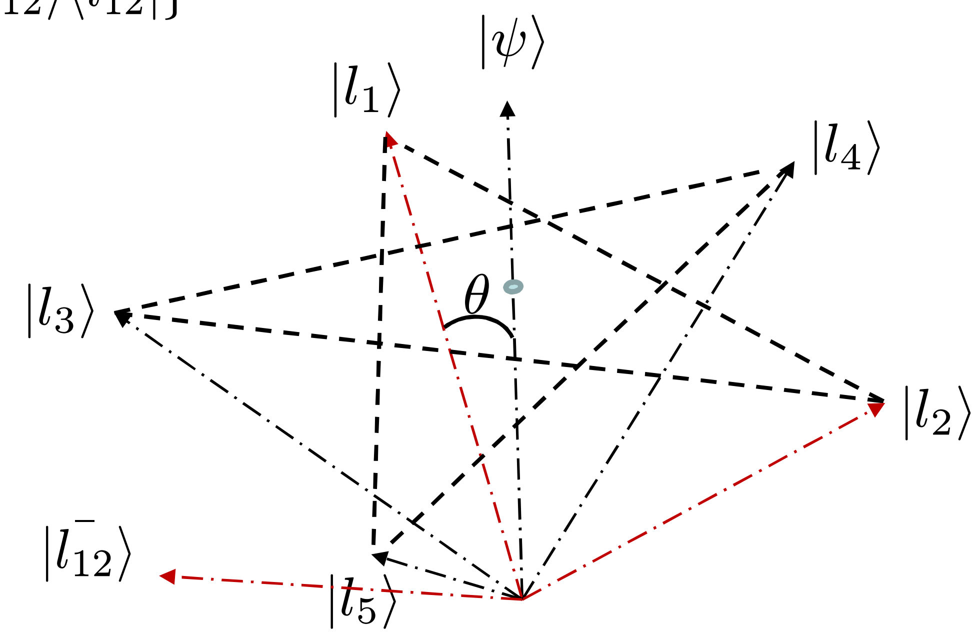
Consider measuring:

$$\{|l_1\rangle\langle l_1| \, , \, I - |l_1\rangle\langle l_1|\}$$

$$\{|l_2\rangle\langle l_2| \, , \, I - |l_2\rangle\langle l_2|\}$$

$$\cos^2\theta = \frac{1}{\sqrt{5}}$$

Equivalently: $\{|l_1\rangle\langle l_1| \, , \, |l_2\rangle\langle l_2| \, , \, |\bar{l_{12}}\rangle\langle \bar{l_{12}}|\}$



Klyachko's example

Consider measuring:

$$\{|l_1\rangle\langle l_1| \ , \ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2| \ , \ I - |l_2\rangle\langle l_2|\}$$

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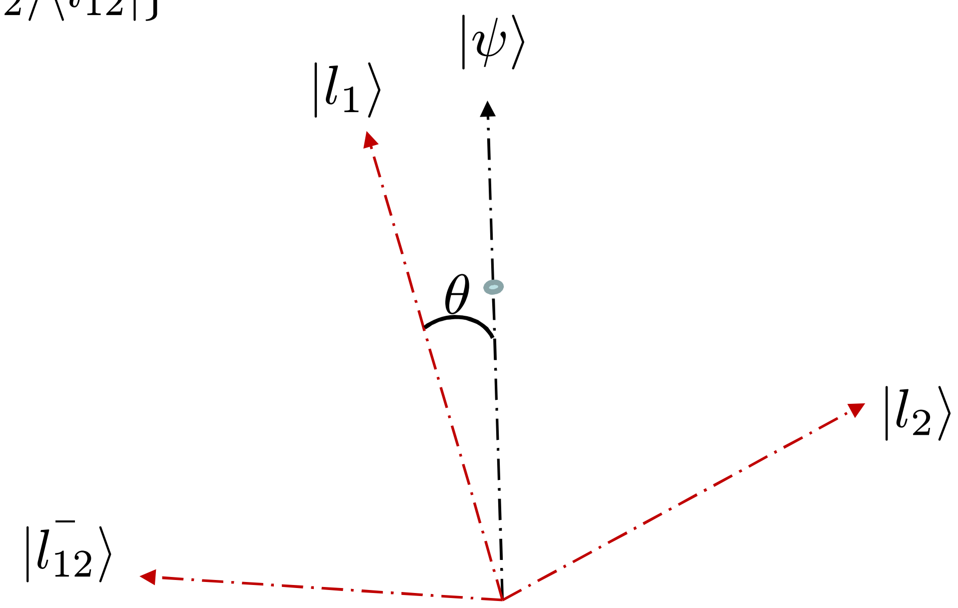

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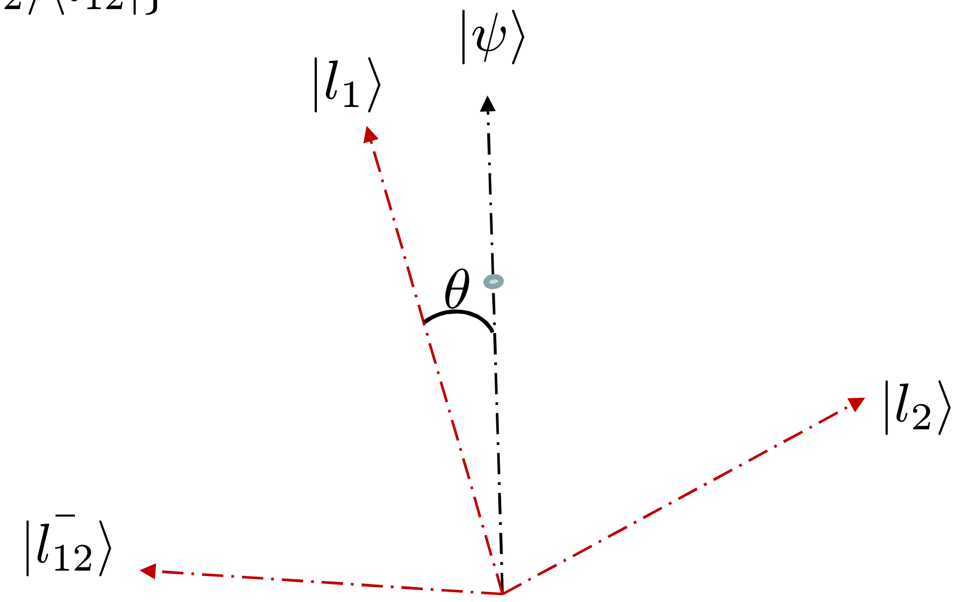

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$$\{|l_2\rangle\langle l_2| \;,\; I - |l_2\rangle\langle l_2|\}$$

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$$\{|l_1\rangle\langle l_1|\ ,\ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2|\ ,\ I - |l_2\rangle\langle l_2|\}$$

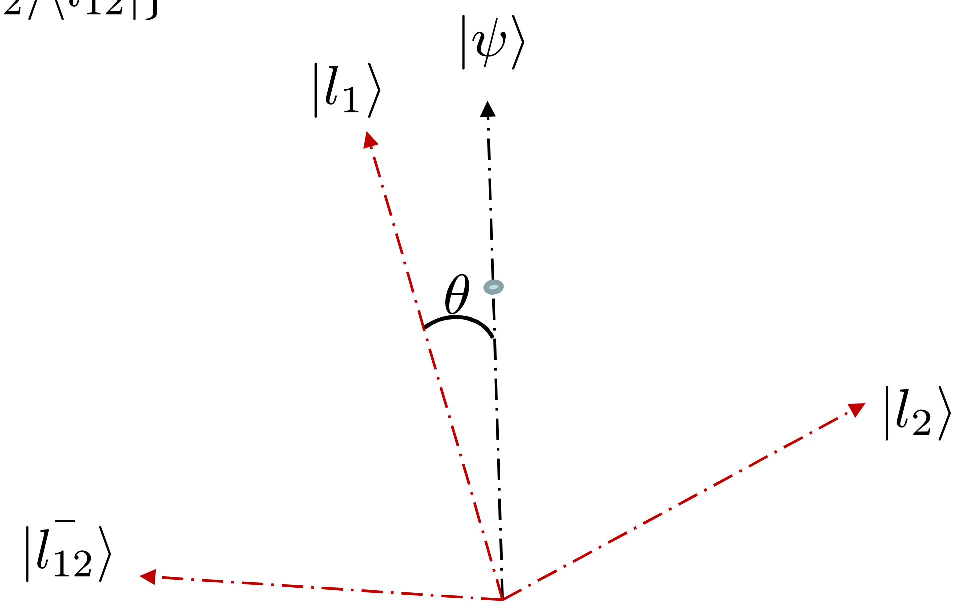
prob. $|\langle\psi|l_1\rangle|^2 = \frac{1}{\sqrt{5}}$

$$\{|l_1\rangle\langle l_1|\ ,\ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2|\ ,\ I - |l_2\rangle\langle l_2|\}$$

prob. $|\langle\psi|l_2\rangle|^2 = \frac{1}{\sqrt{5}}$

$$\{|l_1\rangle\langle l_1|\ ,\ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2|\ ,\ I - |l_2\rangle\langle l_2|\}$$

prob. $|\langle\psi|\bar{l_{12}}\rangle|^2 = 1 - \frac{2}{\sqrt{5}}$



Klyachko's example

Consider measuring:

$$\{|l_1\rangle\langle l_1| \ , \ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2| \ , \ I - |l_2\rangle\langle l_2|\}$$

$$\cos^2\theta = \frac{1}{\sqrt{5}}$$

Equivalently: $\{|l_1\rangle\langle l_1| \ , \ |l_2\rangle\langle l_2| \ , \ |\bar{l}_{12}\rangle\langle \bar{l}_{12}|\}$

$$\{|l_1\rangle\langle l_1| \ , \ I - |l_1\rangle\langle l_1|\}$$
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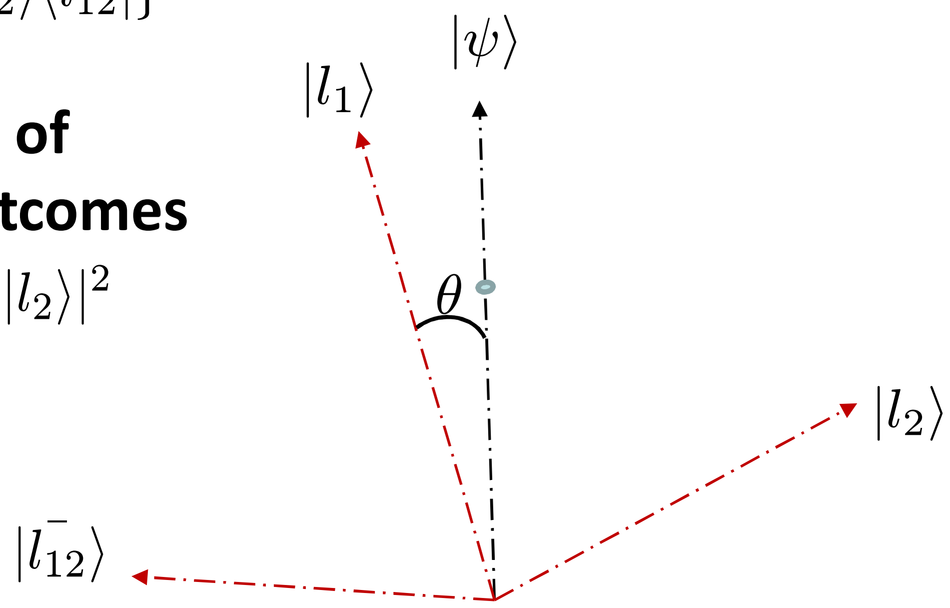
$$\{|l_2\rangle\langle l_2| \ , \ I - |l_2\rangle\langle l_2|\}$$

Probability of anticorreled outcomes

$$|\langle\psi|l_1\rangle|^2 + |\langle\psi|l_2\rangle|^2 = \frac{2}{\sqrt{5}}$$

$$\{|l_1\rangle\langle l_1| \ , \ I - |l_1\rangle\langle l_1|\}$$
$$\{|l_2\rangle\langle l_2| \ , \ I - |l_2\rangle\langle l_2|\}$$

$$\text{prob.}|\langle\psi|\bar{l}_{12}\rangle|^2 = 1 - \frac{2}{\sqrt{5}}$$



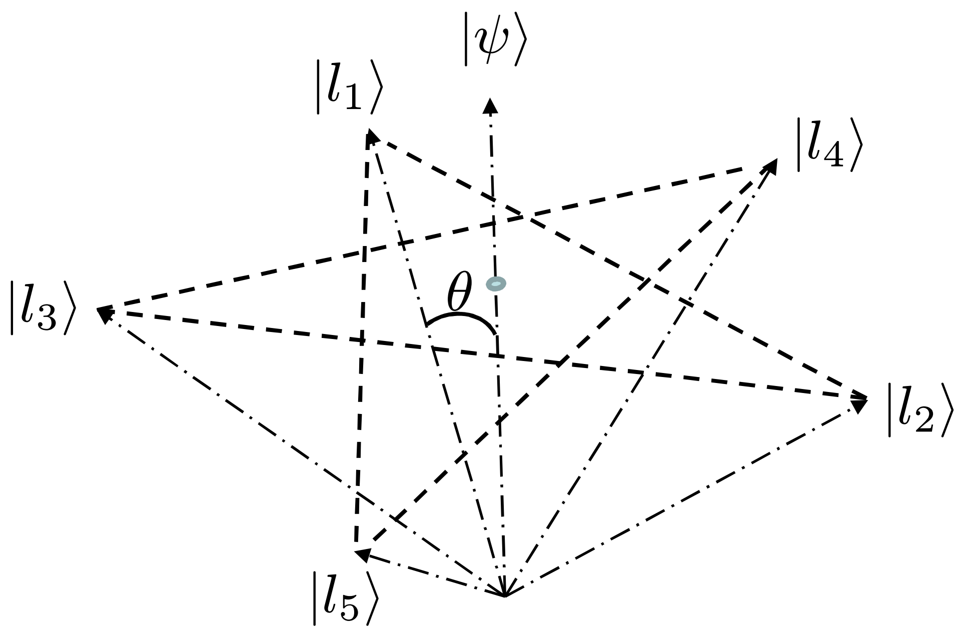
Klyachko's example

$$\cos^2 \theta = \frac{1}{\sqrt{5}}$$

Similarly for any pair of measurements…

Probability of anticorreled outcomes

$$= \frac{2}{\sqrt{5}}$$



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Probability of anticorreled outcomes

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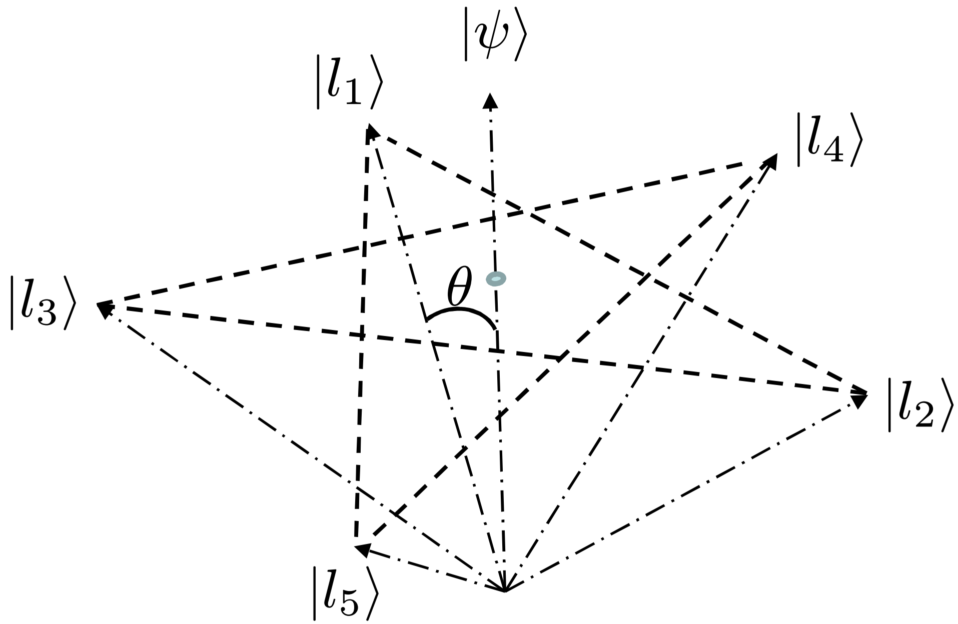


Quantum probability of success

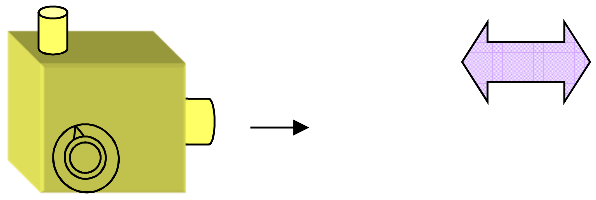
$$p(\text{success}) = \frac{2}{\sqrt{5}} \simeq 0.89 > \frac{4}{5}$$

A generalized notion of noncontextuality for any operational theory

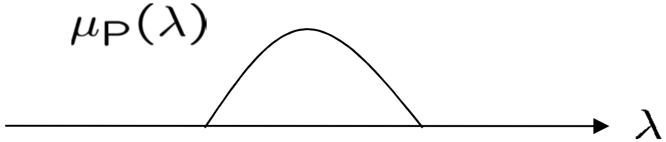
A hidden variable model of an operational theory

Specifies an ontic state space Λ

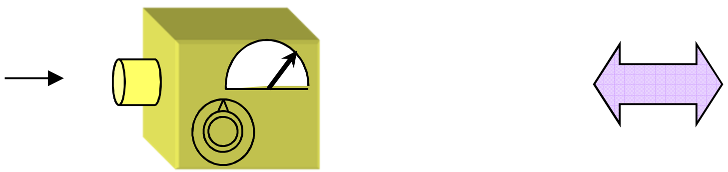
Preparation
P



$$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$$

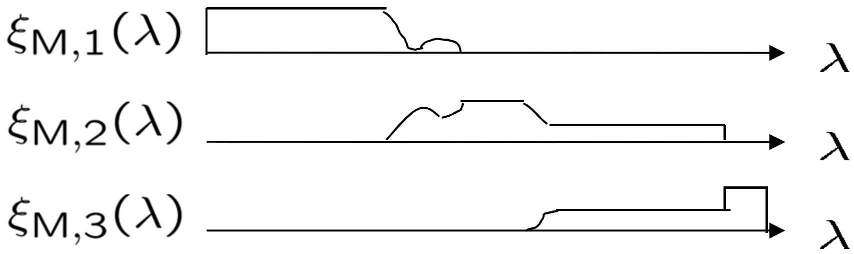


Measurement
M



$$0 \leq \xi_{\mathsf{M},k} \leq 1$$

$$\sum_k \xi_{\mathsf{M},k}(\lambda) = 1 \text{ for all } \lambda$$



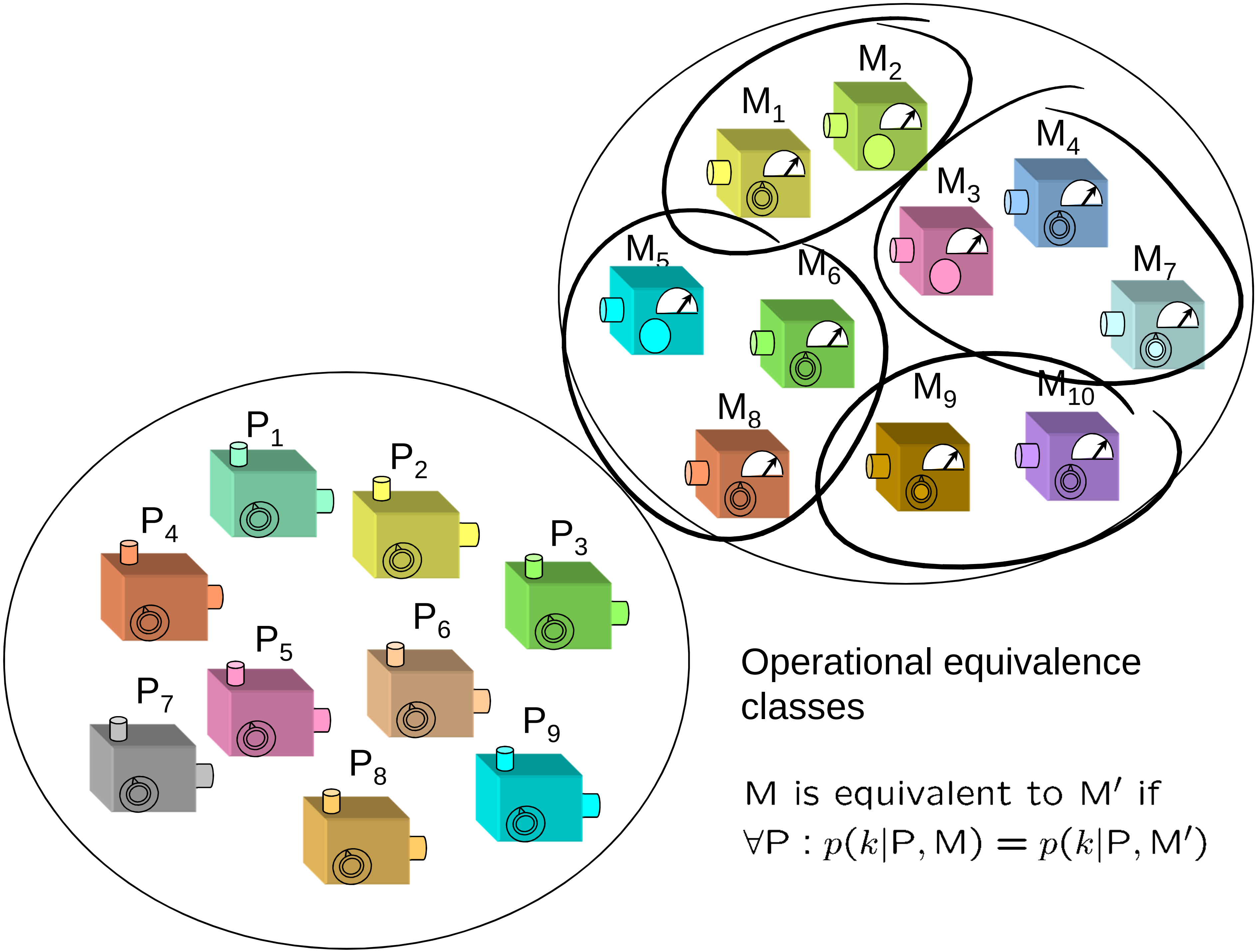
$$p(k|\mathsf{P},\mathsf{M}) = \int d\lambda \; \xi_{\mathsf{M},k}(\lambda) \; \mu_{\mathsf{P}}(\lambda)$$

Generalized definition of noncontextuality:

A hidden variable model of an operational theory is noncontextual if

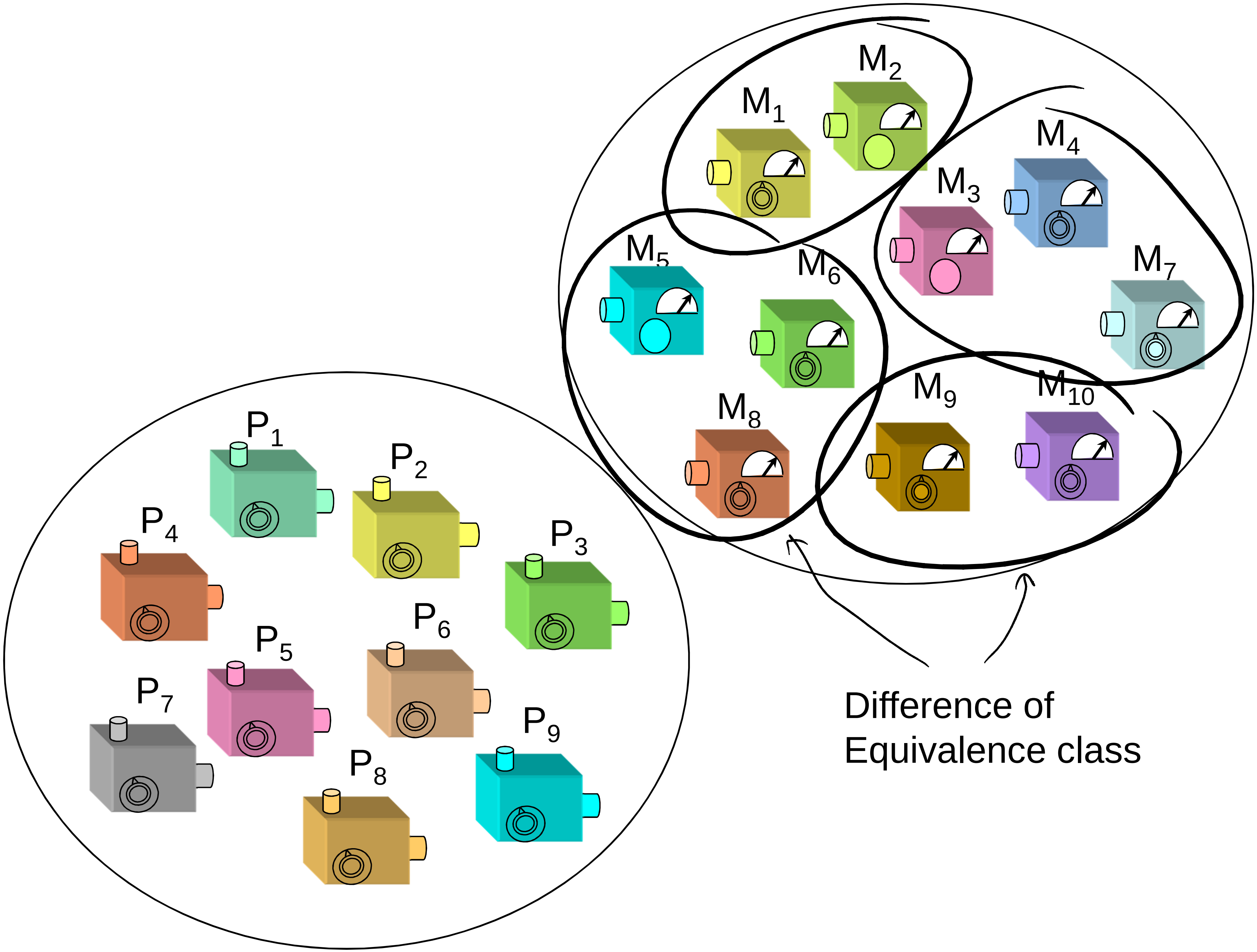
Operational equivalence of two experimental procedures → Equivalent representations in the hidden variable model

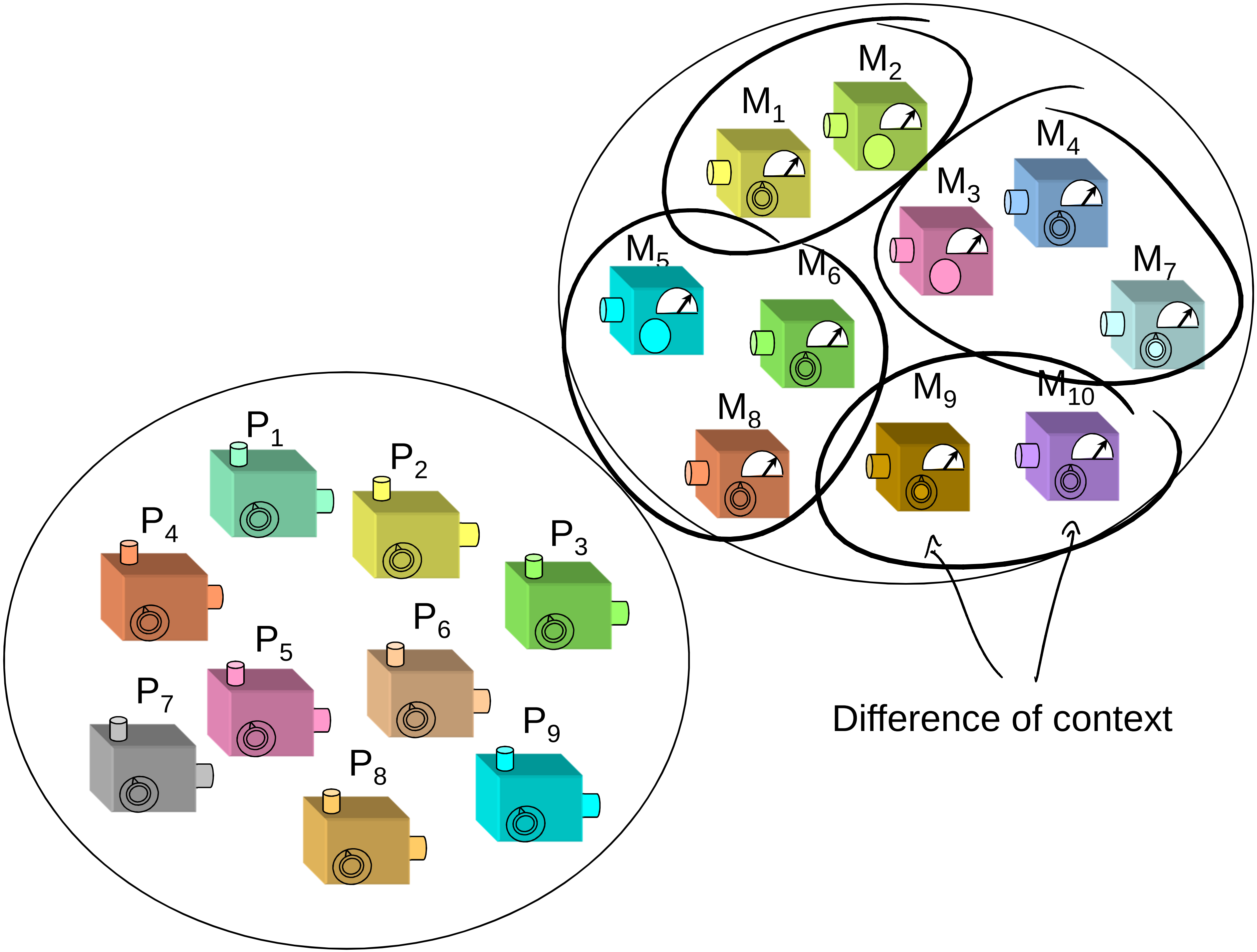
Measurement noncontextuality



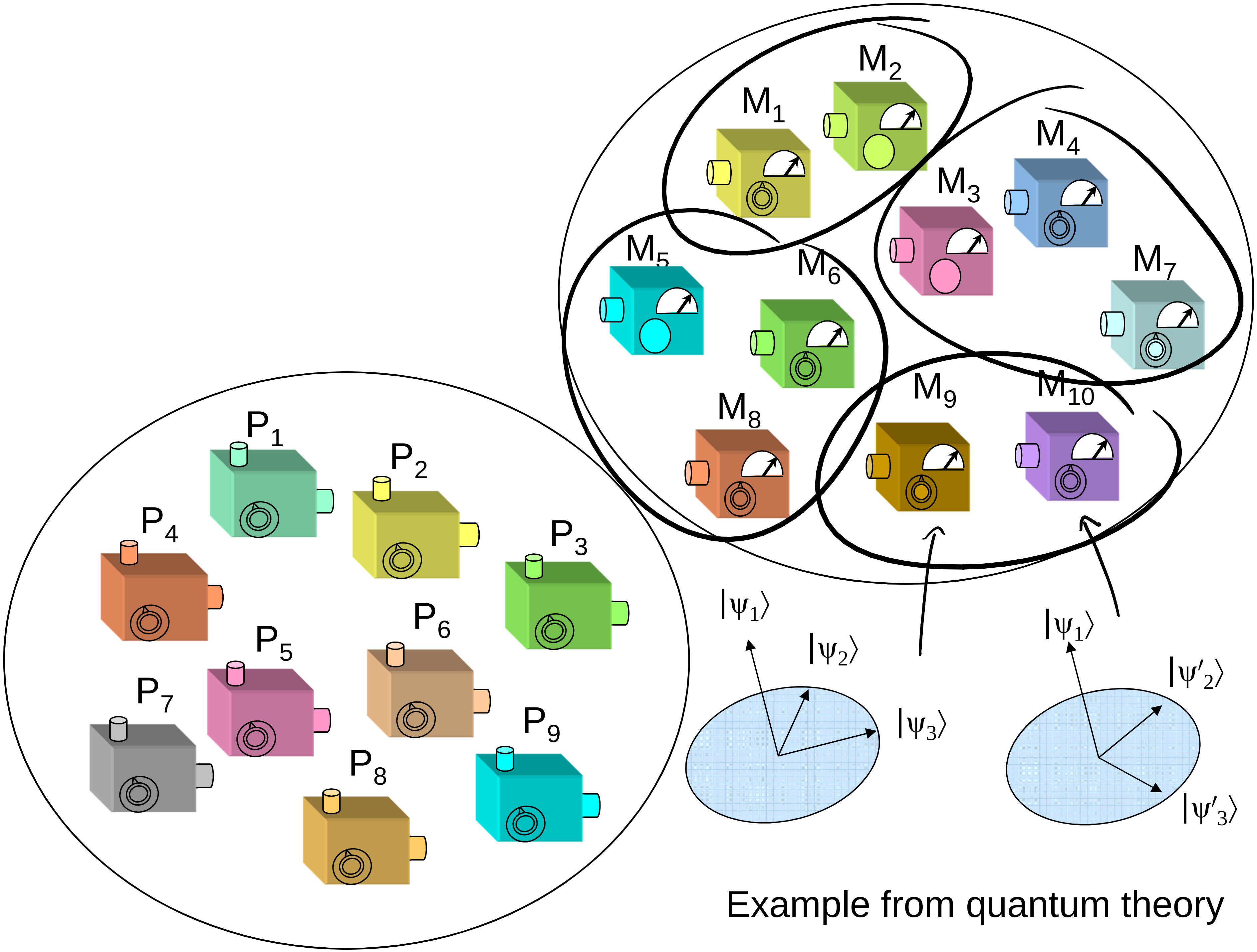
Operational equivalence classes

M is equivalent to M′ if
$\forall \mathsf{P} : p(k|\mathsf{P}, \mathsf{M}) = p(k|\mathsf{P}, \mathsf{M}')$

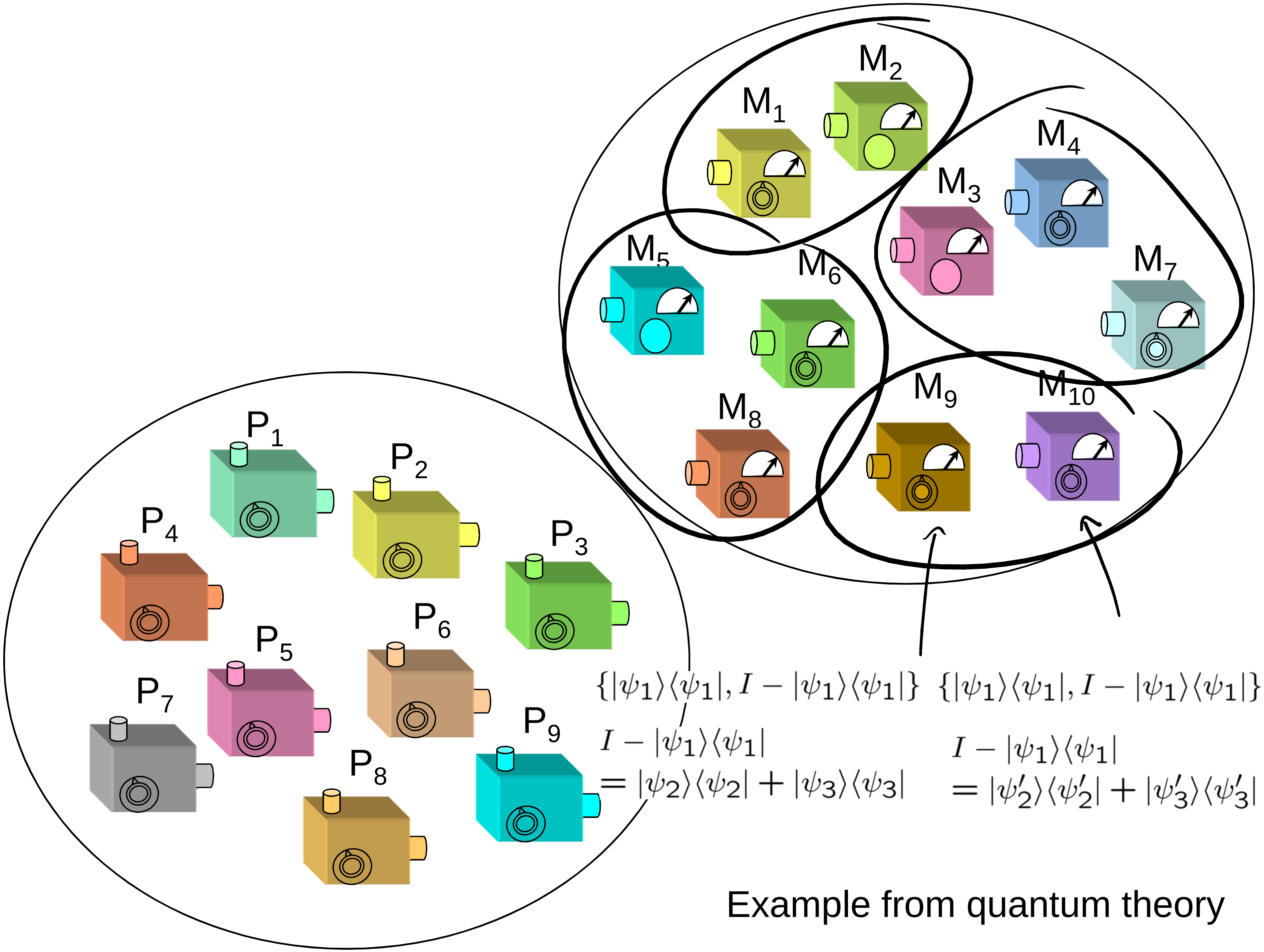

M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9
Difference of
Equivalence class



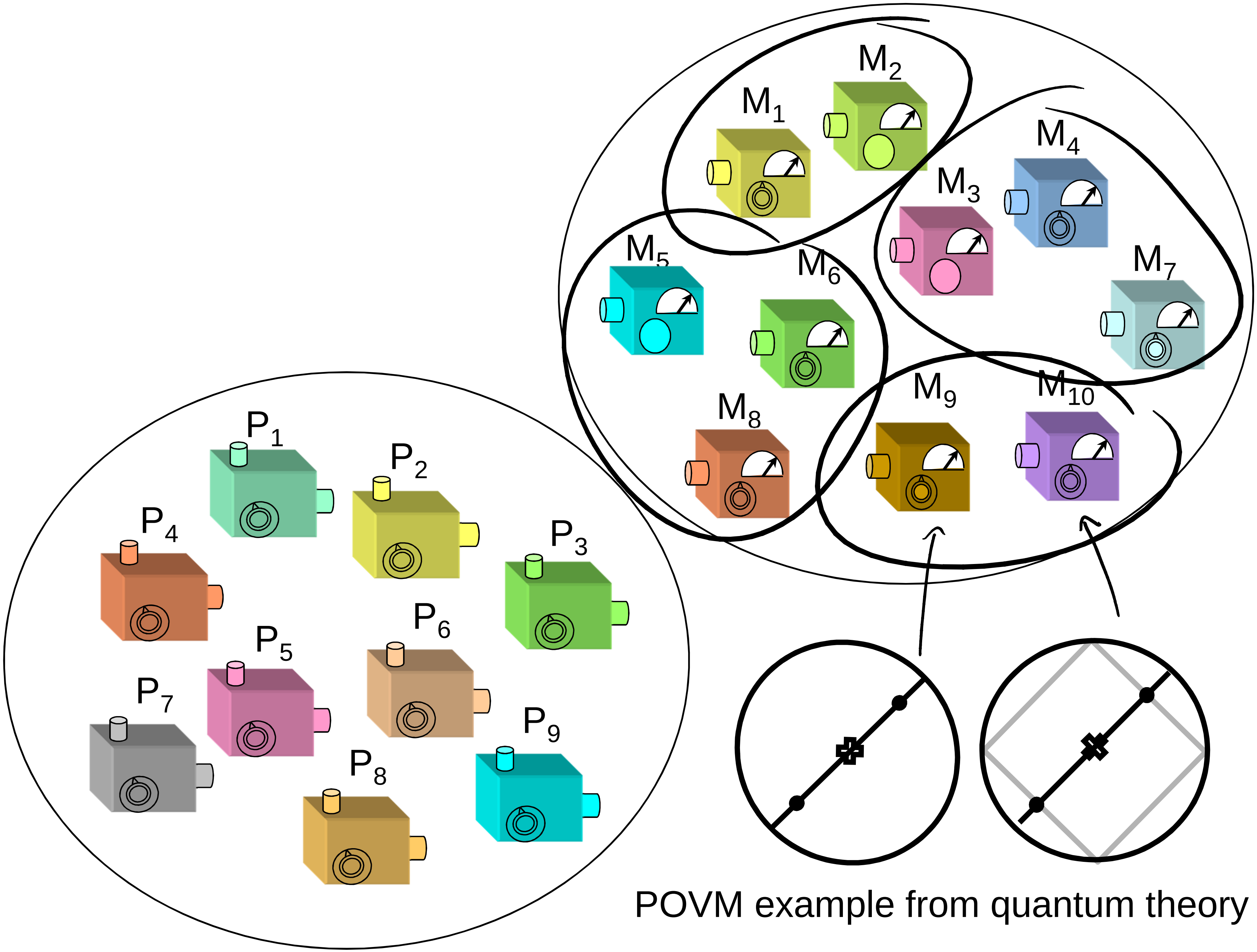
M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
P1
P2
P3
P4
P5
P6
P7
P8
P9
Difference of context



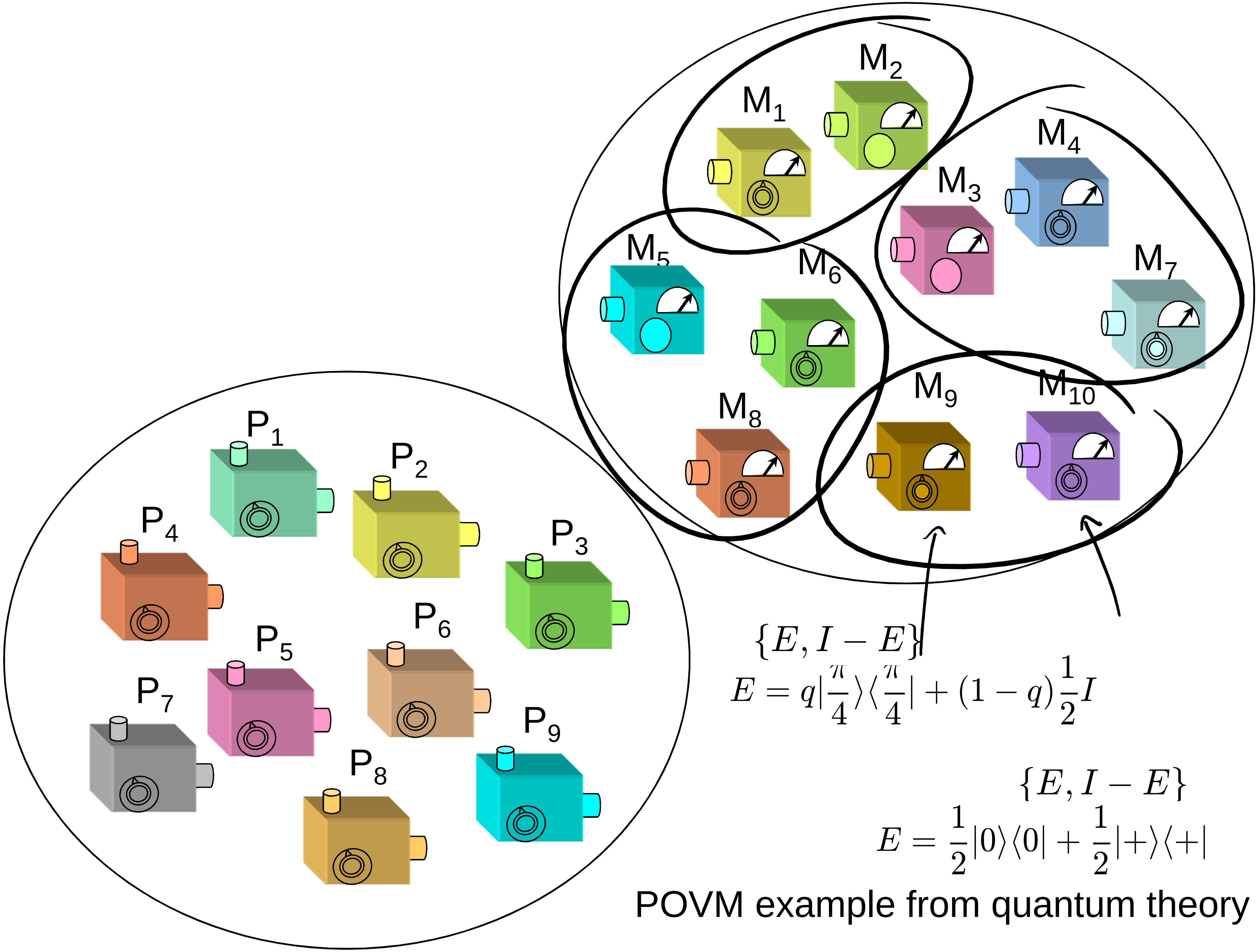
Example from quantum theory



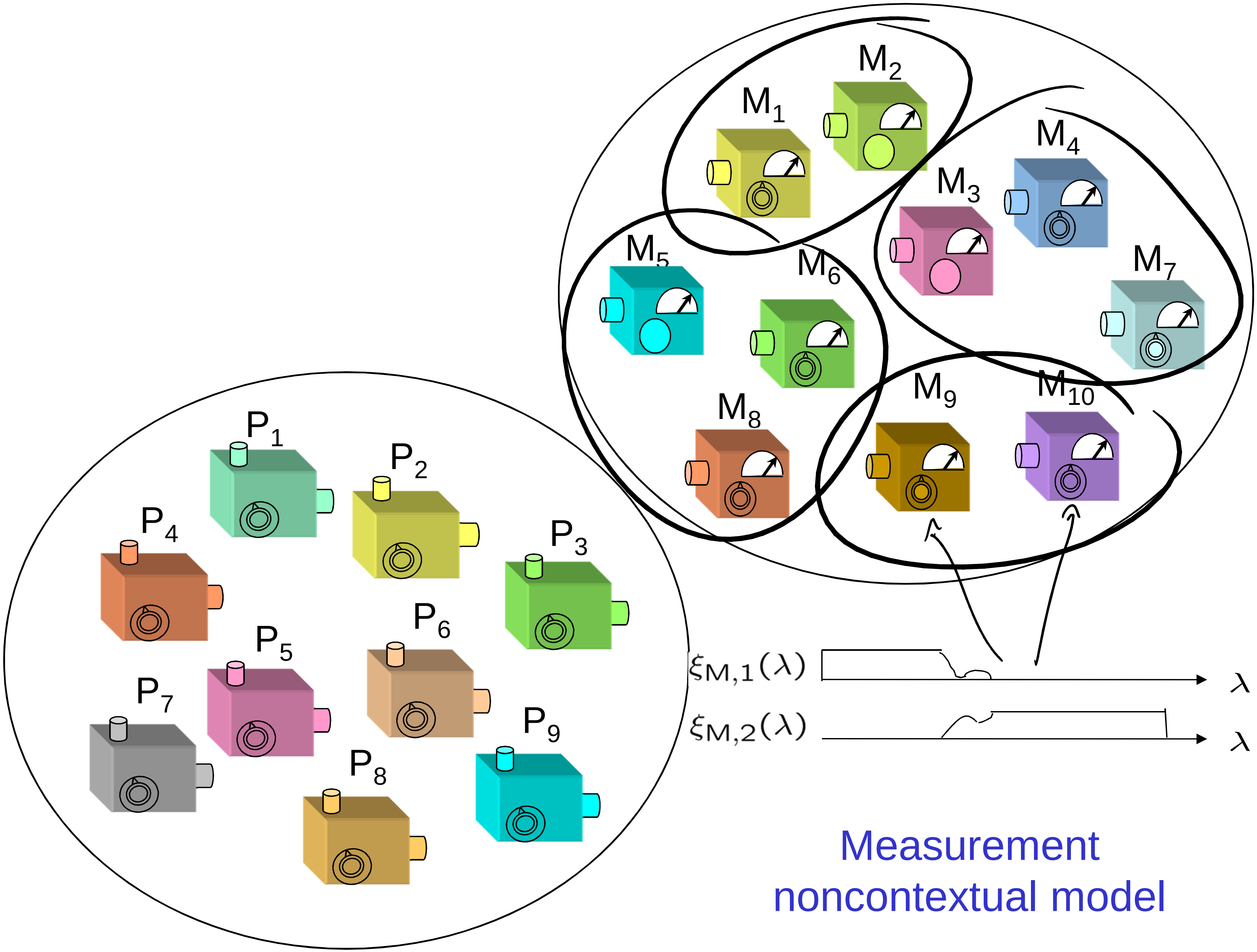
Example from quantum theory



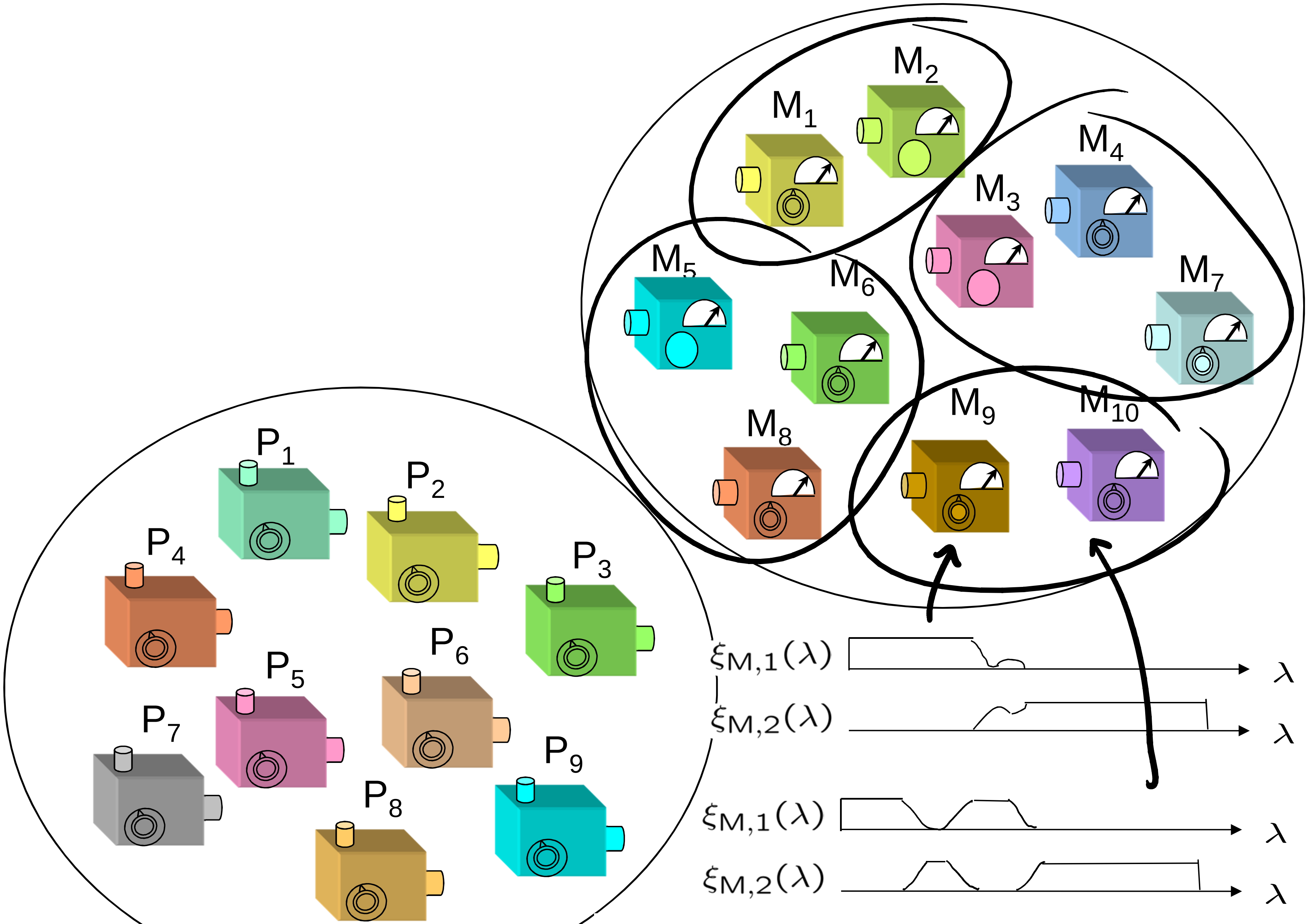
POVM example from quantum theory



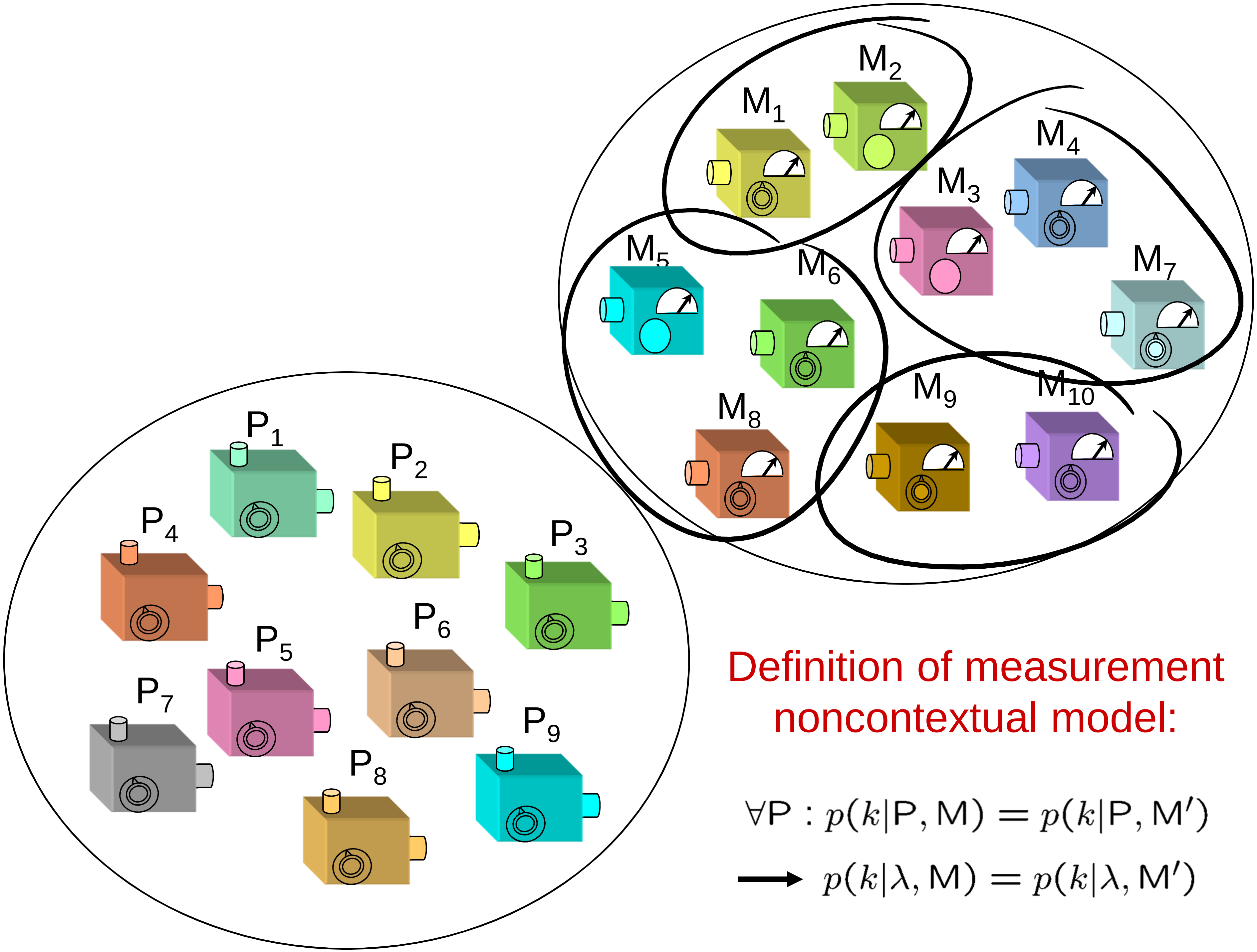
POVM example from quantum theory



Measurement
noncontextual model



Measurement contextual model



Definition of measurement noncontextual model:

$$\forall \mathsf{P} : p(k|\mathsf{P}, \mathsf{M}) = p(k|\mathsf{P}, \mathsf{M}')$$

$$\longrightarrow \; p(k|\lambda, \mathsf{M}) = p(k|\lambda, \mathsf{M}')$$