Solid State Theory
Exercise 8

Exercise 8.1 Bohr–van Leeuwen theorem

Prove the Bohr–van Leeuwen theorem, which states that there is no magnetism in classical physics.
Hint: $\mathcal{H}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)$ is the Hamiltonian of the N-particle system with vanishing external magnetic field. In comparison, the Hamiltonian with applied magnetic field $\boldsymbol{B}$ is then given by $\mathcal{H}(\boldsymbol{p}_1 - e/c\boldsymbol{A}_1, \ldots, \boldsymbol{p}_N - e/c\boldsymbol{A}_N; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)$, where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{A}_i \equiv \boldsymbol{A}(\boldsymbol{q}_i)$. The (thermal average of the) magnetization can be calculated using

$$M = \left\langle -\frac{\partial \mathcal{H}}{\partial B} \right\rangle = \frac{1}{\beta} \frac{\partial \log Z}{\partial B}, \tag{1}$$

with the partition function Z of the system in the magnetic field.

Exercise 8.2 Landau Diamagnetism

Calculate the orbital part of the magnetization (e.g. ignore the Zeeman-term) of the free electron gas in 3D in the limits of low temperature and small external field ($T \to 0,\ B \to 0$). In addition, show that the magnetic susceptibility at $T = 0$ and $B = 0$ is given by

$$\chi = -\frac{1}{3} \frac{m^2}{m^{*2}} \chi_P, \tag{2}$$

where χ_P is the Pauli susceptibility.[1]
Hint: Calculate the free energy,

$$F = N\mu - k_B T \sum_i \ln\left[1 + e^{-(\epsilon_i - \mu)/k_B T}\right], \tag{3}$$

at $T = 0$ to second order in B using the Euler-Maclaurin formula,

$$\sum_0^{n_0} f(n) \approx \int_{-1/2}^{n_0+1/2} f(n)\mathrm{d}n - \frac{1}{24}\left[f'(n_0 + 1/2) - f'(-1/2)\right]. \tag{4}$$

Exercise 8.3 Landau Levels in Graphene

Graphene is defined as a single two-dimensional layer of graphite, the C-atoms are arranged on a two-dimensional honeycomb lattice (cf. exercise 5). The latter is not a Bravais-lattice, but a triangular lattice with a diatomic basis. Consequently, the reciprocal lattice, which is a honeycomb lattice as well, has two inequivalent points called K- and K'-points (see Fig. 1). The two atoms per unit cell create a valence and a conduction band which cross linearly in one point (called the Dirac point) at the K- and K'-points and form the so-called Dirac cones (see Fig. 1). In undoped graphene, the Fermi energy is exactly at the Dirac point.

[1] The Pauli susceptibility $\chi_P = \mu_B^2 \rho(\epsilon_F)$ (at $T = 0$) is a concequence of the Zeeman energy-term in the Hamiltonian for particles with non-zero spin; $\mu_B = \frac{e\hbar}{2mc}$; ρ is the density of states (including the spin degeneracy).

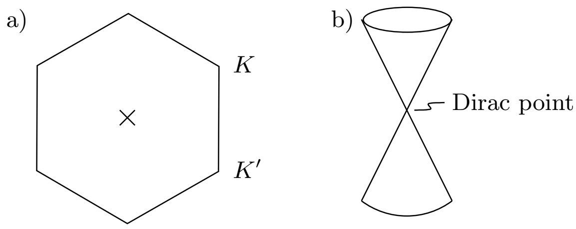

Figure 1: a) First Brillouin zone of graphene with K- and K'-points. b) Band structure of graphene at the K- and K'-points: the Dirac cones.

To a good approximation, the spectrum in graphene is linear at the Fermi energy and described by the Hamiltonian[2]

$$\hat{\mathcal{H}}_{\text{eff}} = v_F \begin{pmatrix} \hat{p}_x\sigma_x + \hat{p}_y\sigma_y & \mathbb{0} \\ \mathbb{0} & \hat{p}_x\sigma_x - \hat{p}_y\sigma_y \end{pmatrix} , \tag{5}$$

where the Pauli matrices σ act on a pseudo-spin[3] and the two parts refer to the inequivalent points K and K'.

a) Using the Peierls-substitution $\boldsymbol{p} \to \boldsymbol{p} - (e/c)\boldsymbol{A}$, find the Landau levels in graphene for a magnetic field perpendicular to the plane (ignore the Zeeman-term).

   Hint: As the 2 parts of the effective Hamiltonian Eq. (5) are independent on each other, it is enough to consider only the first part $\mathcal{H}_K = v_F(p_x\sigma_x + p_y\sigma_y)$; the second part has the same spectrum. Take the “square” of the Schrödinger equation. Note that not only the σ_μ have non-trivial (anti-)commutation relations but also $\boldsymbol{p}$ and $\boldsymbol{r}$ do not commute.

b) Determine the degeneracy of the Landau levels.

c)* Will the magnetization of graphene oscillate when changing the magnetic field? What is the dependence of the ground state energy and the magnetization on a small magnetic field?

Office hour:
Monday, April 23th, 2012 - 10:00 to 12:00 am
HIT G 33.1
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[2]One possible way to come to the Hamiltonian given in Eq. (5) is to recall the Eq. (7) from Solution Sheet 5 with the formula for $\mathcal{H}$ within the tight-binding model on a hexagonal lattice. If you would expand it near the Fermi point K, then substitute there $\hbar\boldsymbol{k}$ by $\hat{\boldsymbol{p}}$, you would (after a rotation of axes x, y) get the first part of the Hamiltonian given in Eq. (5). Similarly one could obtain the second part by expanding near K'.

[3]Recall exercise 5: $\binom{1}{0}$ does correspond to a particle on sublattice A and $\binom{0}{1}$ to a particle on the sublattice B.