

Fermi liquid theory

Concept of Landau's Fermi liquid theory

elementary excitations of interacting Fermions are described by almost independent fermionic quasiparticles

state of Fermi liquid described simply by quasiparticle distribution

Phenomenological Theory by Landau

energy functional:

$$E = E_0 + \sum_{\vec{k}, \sigma} \epsilon_{\sigma}(\vec{k}) \delta n_{\sigma}(\vec{k}) + \frac{1}{2\Omega} \sum_{\vec{k}, \vec{k}'} \sum_{\sigma, \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\sigma}(\vec{k}) \delta n_{\sigma'}(\vec{k}')$$

deviation from ground state

$$\delta n_{\sigma}(\vec{k}) = n_{\sigma}(\vec{k}) - n_{\sigma}^{(0)}(\vec{k})$$

spin index $\sigma = \pm 1$

ground state distribution

$$n_{\sigma}^{(0)}(\vec{k}) = \Theta(k_F - |\vec{k}|)$$

filled Fermi sea

Fermi liquid theory

$$E = E_0 + \sum_{\vec{k}, \sigma} \epsilon_{\sigma}(\vec{k}) \delta n_{\sigma}(\vec{k}) + \frac{1}{2\Omega} \sum_{\vec{k}, \vec{k}'} \sum_{\sigma, \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\sigma}(\vec{k}) \delta n_{\sigma'}(\vec{k}')$$

effective quasiparticle spectrum:

$$\tilde{\epsilon}_{\sigma}(\vec{k}) = \frac{\delta E}{\delta n_{\sigma}(\vec{k})} = \epsilon_{\sigma}(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\sigma'}(\vec{k}')$$

bare quasiparticle spectrum:

$$\epsilon_{\sigma}(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m^*}$$

effective mass

Fermi velocity:

$$\frac{1}{\hbar} \vec{\nabla}_{\vec{k}} \epsilon_{\sigma}(\vec{k}) \Big|_{k_F} = \vec{v}_F = \frac{\hbar \vec{k}_F}{m^*}$$

density of states at ϵ_F :

$$N(\epsilon_F) = \frac{1}{\Omega} \sum_{\vec{k}, \sigma} \delta(\epsilon_{\sigma}(\vec{k}) - \epsilon_F) = \frac{k_F^2}{\pi^2 \hbar v_F} = \frac{m^* k_F}{\pi^2 \hbar^2}$$

Fermi volume conserved

$$k_F = (3\pi^2 n)^{1/3}$$

Fermi liquid theory

$$E = E_0 + \sum_{\vec{k}, \sigma} \epsilon_{\sigma}(\vec{k}) \delta n_{\sigma}(\vec{k}) + \frac{1}{2\Omega} \sum_{\vec{k}, \vec{k}'} \sum_{\sigma, \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\sigma}(\vec{k}) \delta n_{\sigma'}(\vec{k}')$$

couplings:

$$f_{\sigma\sigma'}(\vec{k}, \vec{k}') = \underbrace{f^s(\hat{k}, \hat{k}')}_{\text{symmetric (charge)}} + \sigma\sigma' \underbrace{f^a(\hat{k}, \hat{k}')}_{\text{antisymmetric (spin)}}$$

spherical symmetry: $f^{s,a}(\hat{k}, \hat{k}') = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos \theta_{\hat{k}, \hat{k}'})$ Legendre Polynomials

Landau parameters:

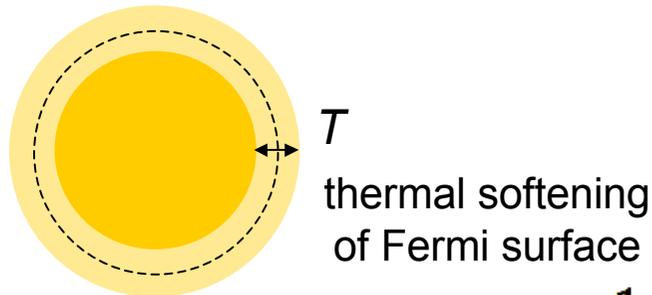
$$F_l^s = N(\epsilon_F) f_l^s \quad \text{charge}$$

$$F_l^a = N(\epsilon_F) f_l^a \quad \text{spin}$$

$$\int_{-1}^{+1} dz P_l(z) P_{l'}(z) = \frac{2\delta_{ll'}}{2l+1}$$

Fermi liquid theory - physical properties

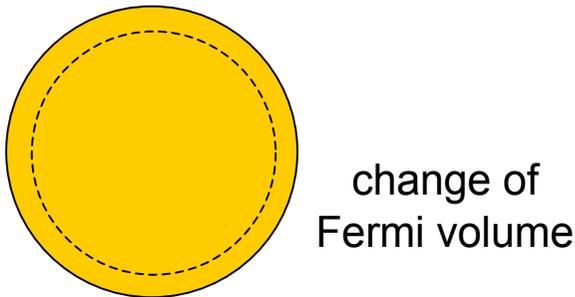
specific heat: $\delta n_\sigma(\vec{k}) = n_\sigma^{(0)}(T, \vec{k}) - n_\sigma^{(0)}(0, \vec{k})$



$$C = \frac{\pi^2 k_B^2 N(\epsilon_F)}{3} T$$

compressibility: $\kappa = -\frac{1}{\Omega} \left. \frac{\partial \Omega}{\partial p} \right|_{T, N}$

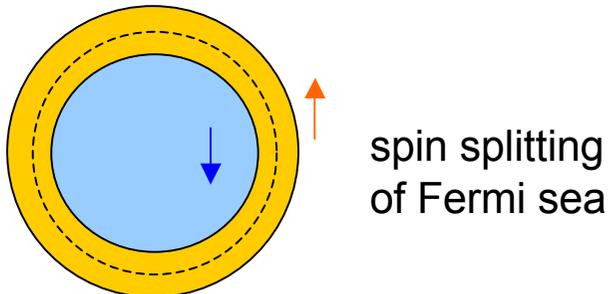
$$\delta \tilde{\epsilon}_\sigma(\vec{k}) = \frac{2}{3} \tilde{\epsilon}_\sigma(\vec{k}) \kappa \delta p$$



$$\kappa = \frac{3}{2n\epsilon_F} \frac{1}{1 + F_0^s} = \frac{1}{n^2} \frac{N(\epsilon_F)}{1 + F_0^s}$$

spin susceptibility: $\chi = \frac{M}{H}$

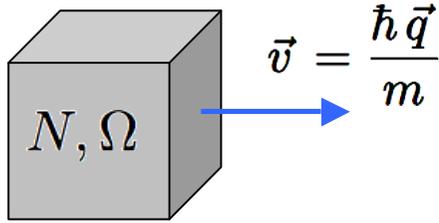
$$\delta \tilde{\epsilon}_\sigma(\vec{k}) = -\tilde{g} \mu_B H \frac{\sigma}{2}$$



$$\chi = \frac{\mu_B^2 N(\epsilon_F)}{1 + F_0^a}$$

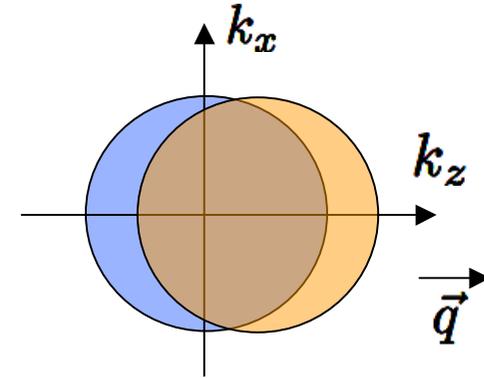
Fermi liquid theory - Gallilei invariance

momentum shift for all particles $\vec{k} \rightarrow \vec{k} + \vec{q}$



current density

$$\vec{j}_{\vec{q}} = \frac{1}{\Omega} \sum_{\vec{k}, \sigma} \vec{v}(\vec{k}) \delta n_{\sigma}(\vec{k})$$



shifted Fermi sea

"bare particle" view

$$\vec{v}(\vec{k}) = \frac{\hbar \vec{k}}{m}$$

quasiparticle view

$$\begin{aligned} \vec{v}(\vec{k}) &= \frac{1}{\hbar} \vec{\nabla}_{\vec{k}} \tilde{\epsilon}_{\sigma}(\vec{k}) \\ &= \frac{1}{\hbar} \left(\vec{\nabla}_{\vec{k}} \epsilon_{\sigma}(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} \vec{\nabla}_{\vec{k}} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\sigma}(\vec{k}') \right) \end{aligned}$$

quasiparticle motion

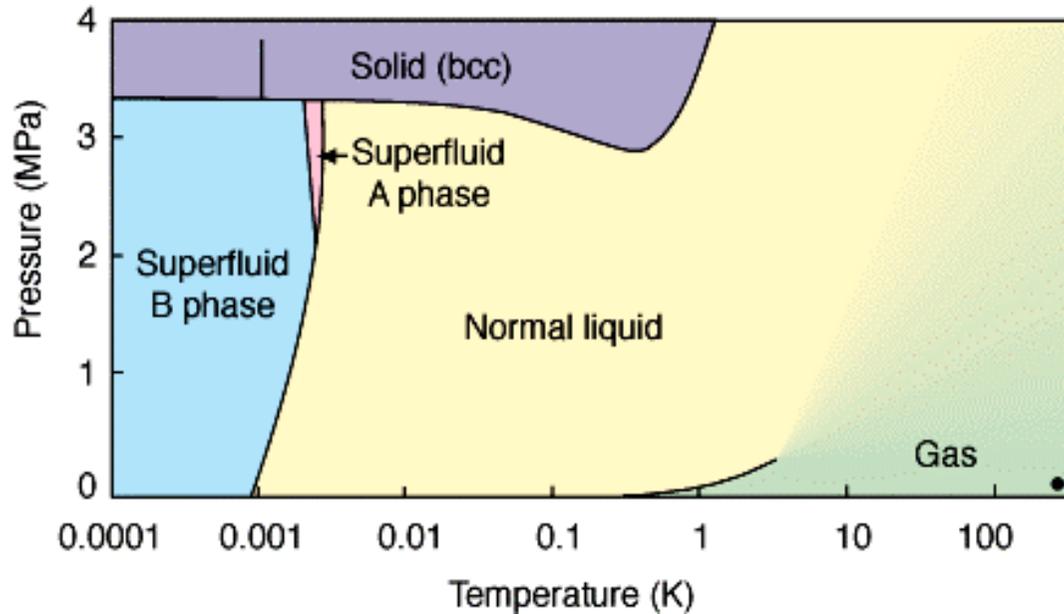
induced motion

$$\frac{\hbar \vec{k}}{m} = \frac{\hbar \vec{k}}{m^*} + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta(\epsilon_{\sigma}(\vec{k}') - \epsilon_F) \frac{\hbar \vec{k}'}{m^*}$$

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^s$$

intrinsic consistency

Fermi liquid theory - ^3He Fermi liquid



strong short-range repulsion



Fermi liquid

longer-ranged attraction



superfluid

pressure	m^*/m	F_0^s	F_0^a	F_1^s	κ/κ_0	χ/χ_0
0	3.0	10.1	-0.52	6.0	0.27	6.3
$< p_c$	6.2	94	-0.74	15.7	0.065	24

↑
enhanced

↑ diminished ↑ enhanced

Hamiltonian

$$\mathcal{H} = \sum_{\vec{k}, s} \epsilon_{\vec{k}} \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} + \int d^3r d^3r' \hat{\Psi}_\uparrow(\vec{r})^\dagger \hat{\Psi}_\downarrow(\vec{r}')^\dagger U \delta(\vec{r} - \vec{r}') \hat{\Psi}_\downarrow(\vec{r}') \hat{\Psi}_\uparrow(\vec{r})$$

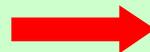
$$= \underbrace{\sum_{\vec{k}, s} \epsilon_{\vec{k}} \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s}}_{\text{kinetic energy}} + \underbrace{\frac{U}{\Omega} \sum_{\vec{k}, \vec{k}', \vec{q}} \hat{c}_{\vec{k}+\vec{q}\uparrow}^\dagger \hat{c}_{\vec{k}'-\vec{q}\downarrow}^\dagger \hat{c}_{\vec{k}'\downarrow} \hat{c}_{\vec{k}\uparrow}}_{\text{contact interaction}}$$

kinetic energy

contact interaction

„perturbation“

$$\epsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}$$

Rayleigh-Schrödinger
perturbation theoryFermi liquid theory
Landau parameters

Rayleigh-Schrödinger

energy: $E = E^{(0)} + E^{(1)} + E^{(2)} + \dots$

$$E^{(0)} = \sum_{\vec{k}, s} \epsilon_{\vec{k}} n_{\vec{k}s} \qquad E^{(1)} = \frac{U}{\Omega} \sum_{\vec{k}, \vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (1 - n_{\vec{k} + \vec{q}\uparrow}) (1 - n_{\vec{k}' - \vec{q}\downarrow})}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k} + \vec{q}} - \epsilon_{\vec{k}' - \vec{q}}}$$

Distribution function
characterizes unperturbed state

$$n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

Rayleigh-Schrödinger

$$n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$E^{(1)} = \frac{U}{\Omega} \sum_{\vec{k}, \vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) + n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{E}^{(1)} = E^{(1)} + \frac{U^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} \approx \frac{\tilde{U}}{\Omega} \sum_{\vec{k}, \vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

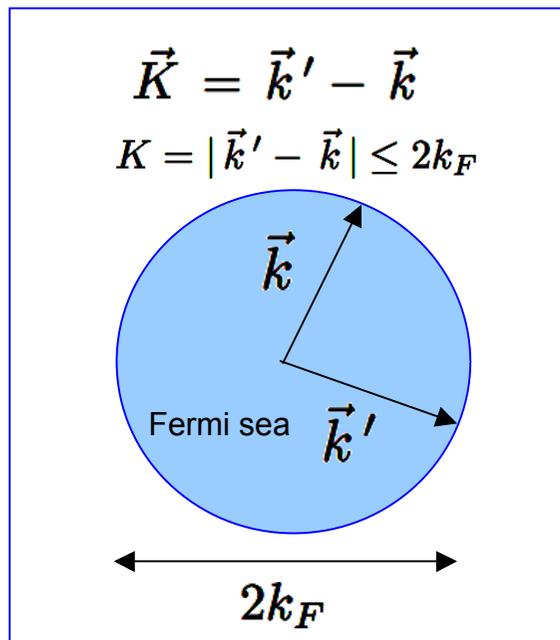
$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

Rayleigh-Schrödinger $n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$

$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\frac{1}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} = \frac{1}{(2\pi)^3} \int_0^{Q_c} dq q^2 \int d\Omega_q \frac{m}{(\vec{k}' - \vec{k}) \cdot \vec{q} - q^2}$$

$$= \frac{m}{(2\pi)^2} \int dq q \int_{-1}^{+1} \frac{d \cos \theta}{K \cos \theta - q}$$



Rayleigh-Schrödinger

$$n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} < U \text{ „screening“}$$

$$\frac{1}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} = \frac{1}{(2\pi)^3} \int_0^{Q_c} dq q^2 \int d\Omega_q \frac{m}{(\vec{k}' - \vec{k}) \cdot \vec{q} - q^2}$$

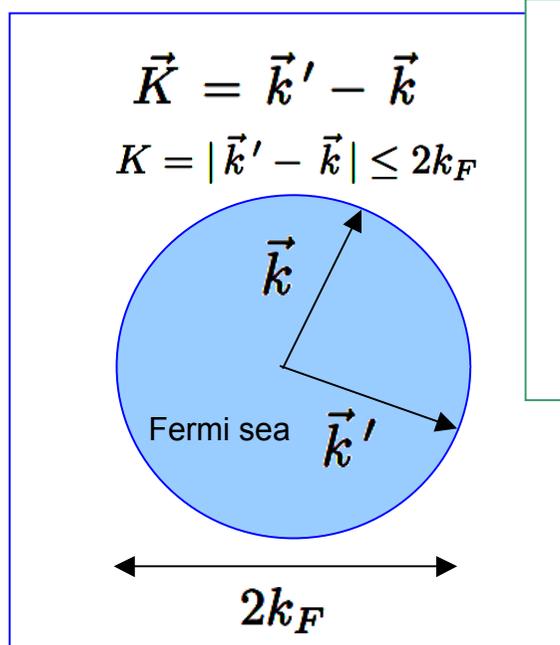
$$= \frac{m}{(2\pi)^2} \int dq q \int_{-1}^{+1} \frac{d \cos \theta}{K \cos \theta - q}$$

$$= \frac{m}{(2\pi)^2} \int_0^{Q_c} dq q \ln \left| \frac{q-K}{q+K} \right|$$

$$= -\frac{m}{(2\pi)^2} \left\{ Q_c + \frac{K^2 - Q_c^2}{2K} \ln \left| \frac{Q_c - K}{Q_c + K} \right| \right\}$$

$$\approx -\frac{2mQ_c}{(2\pi)^2} \left\{ 1 - \frac{K^2}{Q_c^2} + O\left(\frac{K^4}{Q_c^4}\right) \right\} \quad K \ll Q_c$$

weak K-dependence

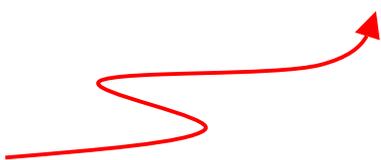


Rayleigh-Schrödinger $n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} - n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) + n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

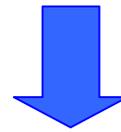
cancels

relabeling $\left\{ \begin{array}{l} \vec{k} \leftrightarrow \vec{k} + \vec{q} \\ \vec{k}' \leftrightarrow \vec{k}' - \vec{q} \end{array} \right\}$



Rayleigh-Schrödinger $n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$

$$E^{(2)} = \frac{U^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} - n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) + n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$



$$\tilde{E}^{(2)} = -\frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow})}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{U} = U + \frac{U^2}{\Omega} \sum_{\vec{q}} \frac{1}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

Rayleigh-Schrödinger $E = E^{(0)} + \tilde{E}^{(1)} + \tilde{E}^{(2)} + \dots$

$$E^{(0)} = \sum_{\vec{k}, s} \epsilon_{\vec{k}} n_{\vec{k}s} \quad \tilde{E}^{(1)} = \frac{\tilde{U}}{\Omega} \sum_{\vec{k}, \vec{k}'} n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow}$$

$$\tilde{E}^{(2)} = -\frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}, \vec{k}', \vec{q}} \frac{n_{\vec{k}\uparrow} n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow})}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

variation: $\tilde{\epsilon}_{\uparrow}(\vec{k}) = \frac{\delta E}{\delta n_{\vec{k}\uparrow}} \quad n_{\vec{k}s} = \langle \hat{c}_{\vec{k}s}^\dagger \hat{c}_{\vec{k}s} \rangle = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$

$$\tilde{\epsilon}_{\uparrow}(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\vec{k}\sigma'}, \quad n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\vec{k}\sigma'}, \quad n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\uparrow}(\vec{k}_F, \vec{k}'_F)$$

$$\frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} n_{\vec{k}+\vec{q}\uparrow} \frac{n_{\vec{k}'-\vec{q}\downarrow} - n_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\xrightarrow{\vec{k}+\vec{q} \rightarrow \vec{k}'_F} \frac{1}{\Omega} \sum_{\vec{k}'_F} n_{\vec{k}'_F\uparrow} \frac{\tilde{U}^2}{\Omega} \sum_{\vec{k}'} \frac{n_{\vec{k}'-\vec{q}\downarrow}^{(0)} - n_{\vec{k}'\downarrow}^{(0)}}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}'-\vec{q}}} \Big|_{\vec{q} = \vec{k}'_F - \vec{k}_F}$$

$$n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s} = -\frac{1}{\Omega} \sum_{\vec{k}'_F} n_{\vec{k}'_F\uparrow} \frac{\tilde{U}^2}{2} \chi_0(\vec{k}'_F - \vec{k}_F)$$

Lindhard function

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\vec{k}\sigma'}, \quad n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\uparrow}(\vec{k}_F, \vec{k}'_F)$$

$$\frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} n_{\vec{k}+\vec{q}\uparrow} \frac{n_{\vec{k}'-\vec{q}\downarrow} - n_{\vec{k}'\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\uparrow}(\vec{k}_F, \vec{k}'_F) = f_{\downarrow\downarrow}(\vec{k}_F, \vec{k}'_F) = \frac{\tilde{U}^2}{2} \chi_0(\vec{k}_F - \vec{k}'_F)$$

$$n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s} = -\frac{1}{\Omega} \sum_{\vec{k}'_F} n_{\vec{k}'_F\uparrow} \frac{\tilde{U}^2}{2} \chi_0(\vec{k}'_F - \vec{k}_F)$$

Lindhard function

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\vec{k}\sigma'}$$

$$n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\downarrow}(\vec{k}_F, \vec{k}'_F)$$

$$\frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} n_{\vec{k}'\downarrow} \frac{n_{\vec{k}+\vec{q}\uparrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$- \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} n_{\vec{k}'-\vec{q}\downarrow} \frac{n_{\vec{k}'\downarrow} - n_{\vec{k}+\vec{q}\uparrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\vec{k}\sigma'}, \quad n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$

$$f_{\uparrow\downarrow}(\vec{k}_F, \vec{k}'_F)$$

$$f_{\uparrow\downarrow}(\vec{k}_F, \vec{k}'_F) = f_{\downarrow\uparrow}(\vec{k}_F, \vec{k}'_F) = \tilde{U} - \frac{\tilde{U}^2}{2} \left\{ 2\tilde{\chi}_0(\vec{k}_F + \vec{k}'_F) - \chi_0(\vec{k}_F - \vec{k}'_F) \right\}$$

$$\chi_0(\vec{k}'_F - \vec{k}_F) = \frac{1}{\Omega} \sum_{\vec{k}'} \frac{n_{\vec{k}'-\vec{q}\downarrow}^{(0)} - n_{\vec{k}'\downarrow}^{(0)}}{\epsilon_{\vec{k}'} - \epsilon_{\vec{k}'-\vec{q}}} \quad \tilde{\chi}_0(\vec{q}) = \frac{1}{\Omega} \sum_{\vec{k}'} \frac{n_{\vec{k}'+\vec{q}\uparrow}^{(0)} + n_{\vec{k}'\downarrow}^{(0)}}{2\epsilon_F - \epsilon_{\vec{k}'+\vec{q}} - \epsilon_{\vec{k}'}}$$

$$\tilde{\epsilon}_\sigma(\vec{k}) = \epsilon_\sigma(\vec{k}) + \frac{1}{\Omega} \sum_{\vec{k}', \sigma'} f_{\sigma\sigma'}(\vec{k}, \vec{k}') \delta n_{\vec{k}\sigma'}, \quad n_{\vec{k}s} = n_{\vec{k}s}^{(0)} + \delta n_{\vec{k}s}$$

$$\tilde{\epsilon}_\uparrow(\vec{k}) = \epsilon_{\vec{k}} + \frac{\tilde{U}}{\Omega} \sum_{\vec{k}'} n_{\vec{k}'\downarrow} - \frac{\tilde{U}^2}{\Omega^2} \sum_{\vec{k}', \vec{q}} \frac{n_{\vec{k}'\downarrow} (n_{\vec{k}+\vec{q}\uparrow} + n_{\vec{k}'-\vec{q}\downarrow}) - n_{\vec{k}+\vec{q}\uparrow} n_{\vec{k}'-\vec{q}\downarrow}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}}$$



$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{U}}{2} \left[\left\{ 1 + \frac{\tilde{U}N(\epsilon_F)}{2} \left(2 + \frac{\cos\theta}{2\sin(\theta/2)} \ln \frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)} \right) \right\} \delta_{\sigma\sigma'} - \left\{ 1 + \frac{\tilde{U}N(\epsilon_F)}{2} \left(1 - \frac{\sin(\theta/2)}{2} \ln \frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)} \right) \right\} \sigma\sigma' \right]$$

Landau parameters:

$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{U}}{2} \left[\left\{ 1 + \frac{\tilde{U}N(\epsilon_F)}{2} \left(2 + \frac{\cos\theta}{2\sin(\theta/2)} \ln \frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)} \right) \right\} \delta_{\sigma\sigma'} - \left\{ 1 + \frac{\tilde{U}N(\epsilon_F)}{2} \left(1 - \frac{\sin(\theta/2)}{2} \ln \frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)} \right) \right\} \sigma\sigma' \right]$$

$$f_{\sigma\sigma'}(\vec{k}, \vec{k}') = f^s(\hat{k}, \hat{k}') + \sigma\sigma' f^a(\hat{k}, \hat{k}')$$

$$f^{s,a}(\hat{k}, \hat{k}') = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos\theta_{\hat{k}, \hat{k}'})$$

$$N(\epsilon_F) f_l^{s,a} = F_l^{s,a}$$

Landau parameters:

$$f_{\sigma\sigma'}(\theta) = \frac{\tilde{U}}{2} \left[\left\{ 1 + \frac{\tilde{U}N(\epsilon_F)}{2} \left(2 + \frac{\cos\theta}{2\sin(\theta/2)} \ln \frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)} \right) \right\} \delta_{\sigma\sigma'} - \left\{ 1 + \frac{\tilde{U}N(\epsilon_F)}{2} \left(1 - \frac{\sin(\theta/2)}{2} \ln \frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)} \right) \right\} \sigma\sigma' \right]$$

$$f_{\sigma\sigma'}(\vec{k}, \vec{k}') = \frac{\tilde{u} = \tilde{U}N(\epsilon_F) > 0}{\tilde{u} = \tilde{U}N(\epsilon_F) > 0} \quad \text{repulsive interaction}$$

$$f^{s,a}(\hat{k}, \hat{k}') = F_0^s = \tilde{u} \left\{ 1 + \tilde{u} \left(1 + \frac{1}{6}(2 + \ln 2) \right) \right\} = \tilde{u} + 1.449 \tilde{u}^2 > 0$$

$$N(\epsilon_F) f_l^{s,a} = F_0^a = -\tilde{u} \left\{ 1 + \tilde{u} \left(1 - \frac{2}{3}(1 - \ln 2) \right) \right\} = -\tilde{u} - 0.895 \tilde{u}^2 < 0$$

$$F_1^s = \tilde{u}^2 \frac{2}{15} (7\ln 2 - 1) = 0.514 \tilde{u}^2 > 0$$

$$\underline{\tilde{u} = \tilde{U} N(\epsilon_F) > 0} \quad \text{repulsive interaction}$$

$$F_0^s = \tilde{u} \left\{ 1 + \tilde{u} \left(1 + \frac{1}{6}(2 + \ln 2) \right) \right\} = \tilde{u} + 1.449 \tilde{u}^2 > 0$$

$$F_0^a = -\tilde{u} \left\{ 1 + \tilde{u} \left(1 - \frac{2}{3}(1 - \ln 2) \right) \right\} = -\tilde{u} - 0.895 \tilde{u}^2 < 0$$

$$F_1^s = \tilde{u}^2 \frac{2}{15} (7 \ln 2 - 1) = 0.514 \tilde{u}^2 > 0$$

$$\frac{\kappa}{\kappa_0} = \frac{m^*}{m} \frac{1}{1 + F_0^s} < 1$$

less compressible

$$\frac{\chi}{\chi_0} = \frac{m^*}{m} \frac{1}{1 + F_0^a} > 1$$

more spin
polarizable

ferromagnetism

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^s > 1$$

higher
effective mass

distribution function

ground state $|\Psi\rangle = |\Psi^{(0)}\rangle + |\Psi^{(1)}\rangle + \dots$

$$|\Psi^{(0)}\rangle = |\Psi_0\rangle$$

$$|\Psi^{(1)}\rangle = \frac{U}{\Omega} \sum_{\vec{k}, \vec{k}', \vec{q}} \sum_{s, s'} \frac{\hat{c}_{\vec{k}+\vec{q}, s}^\dagger \hat{c}_{\vec{k}', s'}^\dagger \hat{c}_{\vec{k}', s'} \hat{c}_{\vec{k}, s}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}'} - \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}'-\vec{q}}} |\Psi_0\rangle$$

$$\langle \hat{n}_{\vec{k} s} \rangle = \frac{\langle \Psi | \hat{c}_{\vec{k} s}^\dagger \hat{c}_{\vec{k} s} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\langle \Psi^{(0)} | \hat{c}_{\vec{k} s}^\dagger \hat{c}_{\vec{k} s} | \Psi^{(0)} \rangle + \langle \Psi^{(1)} | \hat{c}_{\vec{k} s}^\dagger \hat{c}_{\vec{k} s} | \Psi^{(1)} \rangle}{\langle \Psi | \Psi \rangle}$$

$$\langle \hat{n}_{\vec{k} s} \rangle = \langle \hat{n}_{\vec{k} s} \rangle^{(0)} + \langle \hat{n}_{\vec{k} s} \rangle^{(2)} + \dots$$

distribution function

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \dots \quad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} = \begin{cases} -\frac{U^2}{\Omega^2} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \frac{(1 - n_{\vec{k}_1})(1 - n_{\vec{k}_2})n_{\vec{k}_3}}{(\epsilon_{\vec{k}} + \epsilon_{\vec{k}_3} - \epsilon_{\vec{k}_1} - \epsilon_{\vec{k}_2})^2} \delta_{\vec{k} + \vec{k}_3, \vec{k}_1 + \vec{k}_2} & |\vec{k}| < k_F \\ \frac{U^2}{\Omega^2} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \frac{n_{\vec{k}_1} n_{\vec{k}_2} (1 - n_{\vec{k}_3})}{(\epsilon_{\vec{k}_1} + \epsilon_{\vec{k}_2} - \epsilon_{\vec{k}} - \epsilon_{\vec{k}_3})^2} \delta_{\vec{k} + \vec{k}_3, \vec{k}_1 + \vec{k}_2} & |\vec{k}| > k_F \end{cases}$$

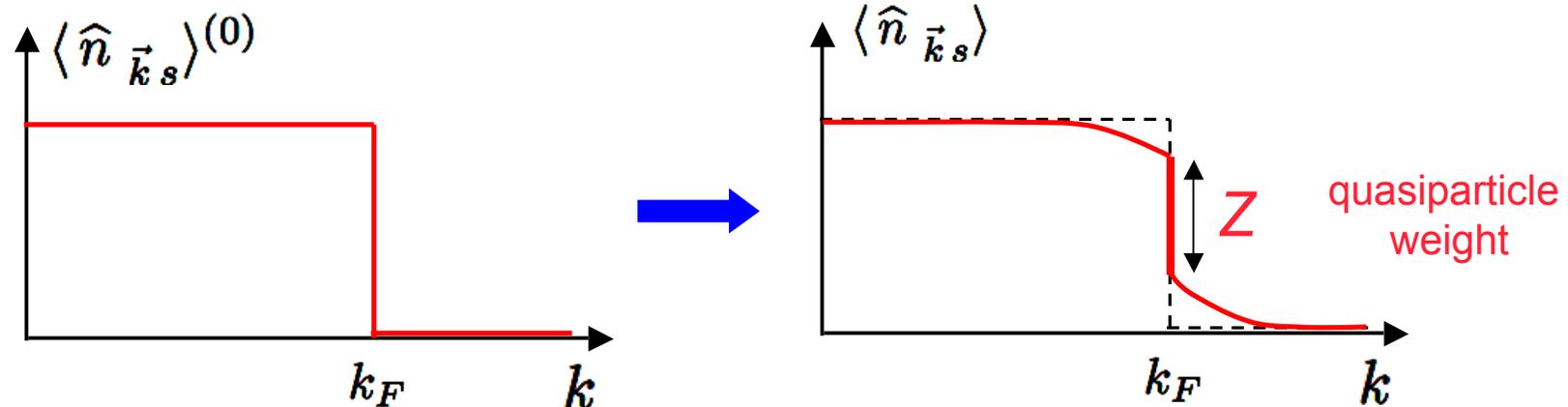
3-dimensional Fermigas

$$\langle \hat{n}_{\vec{k}_{F-}} \rangle - \langle \hat{n}_{\vec{k}_{F+}} \rangle = 1 - \left(\frac{UN(\epsilon_F)}{2} \right)^2 \ln 2$$

distribution function

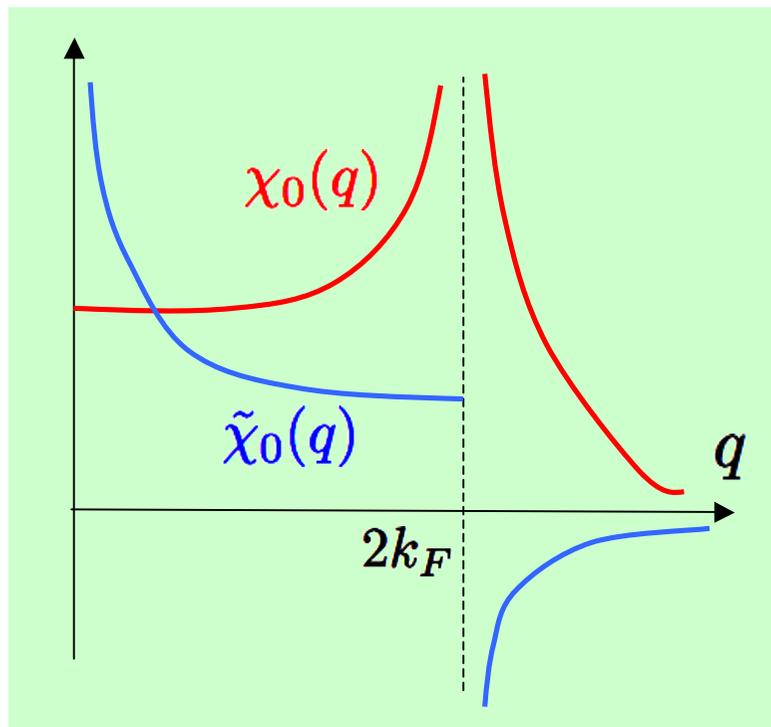
$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \dots \quad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

$$\langle \hat{n}_{\vec{k}_{F-}} \rangle - \langle \hat{n}_{\vec{k}_{F+}} \rangle = 1 - \left(\frac{UN(\epsilon_F)}{2} \right)^2 \ln 2$$

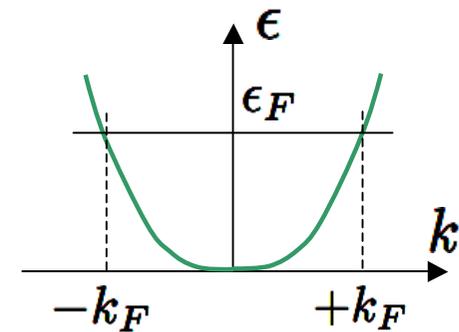


Landau parameters $f_{\sigma\sigma}(k_F, k'_F) = \frac{\tilde{U}^2}{2} \chi_0(k_F - k'_F)$

$$f_{\sigma\bar{\sigma}}(k_F, k'_F) = \tilde{U} - \frac{\tilde{U}^2}{2} \{2\tilde{\chi}_0(k_F + k'_F) - \chi_0(k_F - k'_F)\}$$



$$q = 0, \pm 2k_F$$



$$f_{\sigma\sigma}(\pm k_F, \mp k_F) = \infty$$

$$f_{\sigma\bar{\sigma}}(k_F, \mp k_F) = \infty$$

breakdown of perturbation approach !

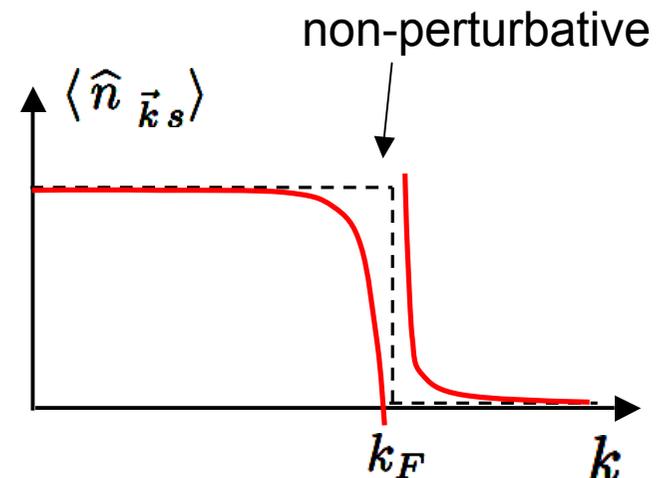
distribution function

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \dots \quad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} = \begin{cases} -\frac{U^2}{\Omega^2} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \frac{(1 - n_{\vec{k}_1})(1 - n_{\vec{k}_2})n_{\vec{k}_3}}{(\epsilon_{\vec{k}} + \epsilon_{\vec{k}_3} - \epsilon_{\vec{k}_1} - \epsilon_{\vec{k}_2})^2} \delta_{\vec{k} + \vec{k}_3, \vec{k}_1 + \vec{k}_2} & |\vec{k}| < k_F \\ \frac{U^2}{\Omega^2} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \frac{n_{\vec{k}_1}n_{\vec{k}_2}(1 - n_{\vec{k}_3})}{(\epsilon_{\vec{k}_1} + \epsilon_{\vec{k}_2} - \epsilon_{\vec{k}} - \epsilon_{\vec{k}_3})^2} \delta_{\vec{k} + \vec{k}_3, \vec{k}_1 + \vec{k}_2} & |\vec{k}| > k_F \end{cases}$$

1-dimensional Fermigas

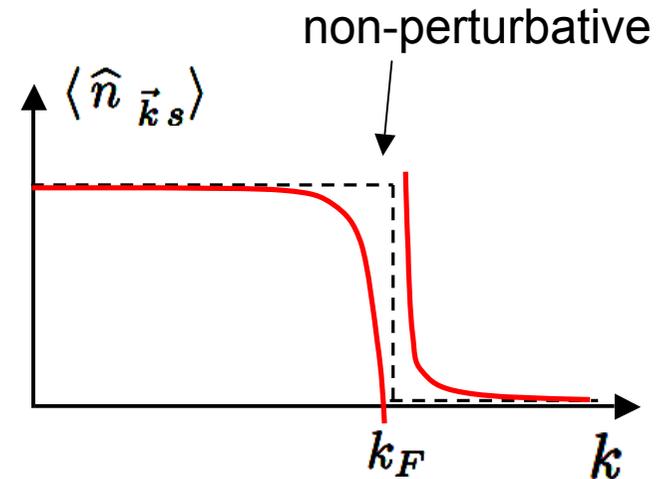
$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} \approx \begin{cases} \frac{1}{8\pi^2} \frac{U^2}{\hbar^2 v_F^2} \ln \frac{k_+}{k - k_F} & k > k_F \\ -\frac{1}{8\pi^2} \frac{U^2}{\hbar^2 v_F^2} \ln \frac{k_-}{k_F - k} & k < k_F \end{cases}$$



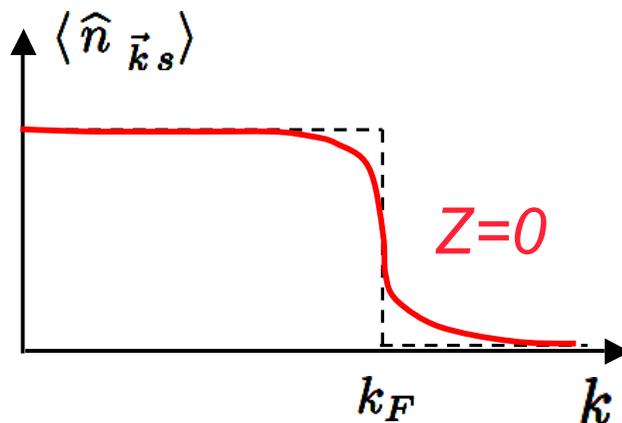
distribution function

$$\langle \hat{n}_{\vec{k}s} \rangle = \langle \hat{n}_{\vec{k}s} \rangle^{(0)} + \langle \hat{n}_{\vec{k}s} \rangle^{(2)} + \dots \quad \langle \hat{n}_{\vec{k}s} \rangle^{(0)} = \Theta(k_F - |\vec{k}|)$$

$$\langle \hat{n}_{\vec{k}s} \rangle^{(2)} \approx \begin{cases} \frac{1}{8\pi^2} \frac{U^2}{\hbar^2 v_F^2} \ln \frac{k_+}{k - k_F} & k > k_F \\ -\frac{1}{8\pi^2} \frac{U^2}{\hbar^2 v_F^2} \ln \frac{k_-}{k_F - k} & k < k_F \end{cases}$$



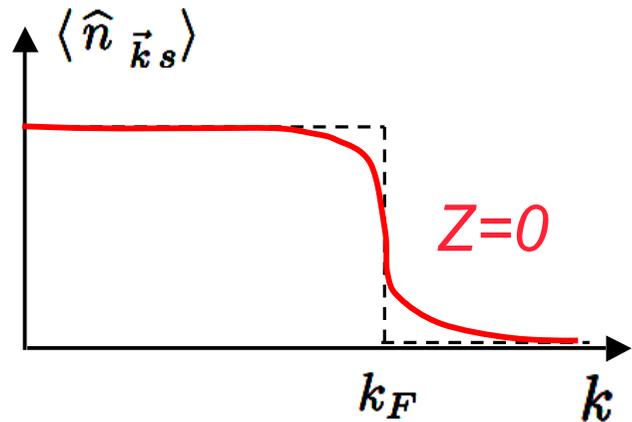
1-dimensional Fermigas



no quasiparticle weight
 ↓
 not a Fermi liquid

Tomonaga
 Luttinger
 liquid

Tomonaga-Luttinger liquid



no quasiparticles



Fermi liquid behavior disappears

excitations:

collective modes (bosonization of Fermions)

separation of charge and spin excitations

