Solid State Theory
Solution Sheet 12

Exercise 12.1 Conductivity tensor

The conductivity tensor in the relaxation time approximation can be derived from Eq. (6.26-27) in the lecture notes

$$\sigma_{\alpha\beta} = -\frac{e^2}{4\pi^3}\int d^3k \frac{\tau(\epsilon_{\boldsymbol{k}})}{1-i\omega\tau(\epsilon_{\boldsymbol{k}})}\frac{\partial f_0(\varepsilon_{\boldsymbol{k}})}{\partial \varepsilon_{\boldsymbol{k}}} v_{\alpha\boldsymbol{k}} v_{\beta\boldsymbol{k}}, \tag{1}$$

with $v_{\alpha\boldsymbol{k}} = \hbar^{-1}\partial_{\boldsymbol{k}_\alpha}\varepsilon_{\boldsymbol{k}}$. For a dispersion relation of the form

$$\varepsilon_{\boldsymbol{k}} = \sum_{\alpha=x,y,z} \frac{\hbar^2 k_\alpha^2}{2m_\alpha} \tag{2}$$

the surfaces of constant energy are ellipsoids and thus we change to corresponding coordinates,

$$\begin{aligned} k_x &= \sqrt{\frac{2m_x\varepsilon}{\hbar^2}}\sin\theta\cos\varphi \\ k_x &= \sqrt{\frac{2m_y\varepsilon}{\hbar^2}}\sin\theta\sin\varphi \\ k_x &= \sqrt{\frac{2m_z\varepsilon}{\hbar^2}}\cos\theta \\ \Rightarrow d^3k &= \sqrt{\frac{2m_xm_ym_z}{\hbar^6}}\sqrt{\varepsilon}\sin\theta d\varepsilon d\theta d\varphi. \end{aligned}$$

The conductivity tensor is thus

$$\sigma_{\alpha\beta} = -\frac{e^2}{4\pi^3}\sqrt{\frac{2m_xm_ym_z}{\hbar^6}}\int d\varepsilon \frac{\tau(\varepsilon)}{1-i\omega\tau(\varepsilon)}\frac{\partial f_0(\varepsilon)}{\partial\varepsilon}\sqrt{\varepsilon}\int d\Omega v_\alpha v_\beta, \tag{3}$$

with $v_\alpha = \hbar k_\alpha/m_\alpha$. If we now use the fact that

$$\int d\Omega v_\alpha v_\beta = \frac{4\pi}{3}\frac{2\varepsilon}{m_\alpha}\delta_{\alpha\beta}, \tag{4}$$

we find for the static case ($\omega = 0$)

$$\sigma_{xx} = -\frac{e^2}{3\pi^2\hbar^3}\sqrt{\frac{8m_ym_z}{m_x}}\int d\varepsilon \frac{\partial f_0(\varepsilon)}{\partial\varepsilon}\varepsilon^{3/2}\tau(\varepsilon) \tag{5}$$

and analogously for σ_{yy} and σ_{zz}. For $T = 0$, this yields

$$\sigma_{\alpha\beta} = \frac{e^2 n\tau(\varepsilon_F)}{m_\alpha}\,\delta_{\alpha\beta}, \tag{6}$$

where we have used that the density of the electrons is given as $n = \sqrt{8m_xm_ym_z}\varepsilon_F^{3/2}/(3\pi^2\hbar^3)$.

Exercise 12.2 Mean free path

In the lecture notes it is shown that

$$\frac{1}{\tau} \propto N(\epsilon_F) n_{\mathrm{imp}} \frac{Z^2}{k_{\mathrm{TF}}^4} \quad \text{and} \quad \rho \propto \frac{1}{n\tau} \tag{7}$$

such that

$$\rho \propto \frac{N(\epsilon_F)}{k_{\mathrm{TF}}^4} \frac{n_{\mathrm{imp}}}{n} Z^2. \tag{8}$$

The first fraction only depends on the Fermi surface which is hardly affected by the impurities and the second fraction does not change much because the doping increases the number of free charge carriers only slightly (very small impurity density). Therefore we can conclude that

$$\rho \propto Z^2. \tag{9}$$

As it was mentioned in the hint we assume that the optimal electronic configuration of the impurity atom valence shell is identical with the valence shell of the surrounding atoms. In this spirit we define the effective charge of the impurity as the number of electrons which have to be removed from the impurity in order to have the same valence shell configuration as the surrounding atoms, because those electrons are fully delocalized and can therefore no longer screen the nucleus of the impurity atom (apart from Thomas-Fermi screening).

A look at the periodic table gives us the electronic configuration, e.g. copper has the valence band configuration $4s^1$. By calculating the difference of the valence electrons between copper and the impurity we find the following table:

Impurity	Electronic configuration	Eff. charge Z
Be	$2s^2$	1
Mg	$3s^2$	1
B	$2s^2 2p^1$	2
Al	$3s^2 3p^1$	2
In	$5s^2 5p^1$	2
Si	$3s^2 3p^2$	3
Ge	$4s^2 4p^2$	3
Sn	$5s^2 5p^2$	3
As	$4s^2 4p^3$	4
Sb	$5s^2 5p^3$	4

Table 1: Electronic configuration and the effective charge of the impurities.

Indeed, by comparing the table above with the table on the exercise sheet, we remark that impurities with identical effective charges have similar residual resistivities in agreement with our considerations.

Exercise 12.3 Magnetoresistance and Hall effect

a) The term in first order for the magnetic field vanishes since

$$\frac{e}{\hbar}(\boldsymbol{v_k} \times \boldsymbol{B}) \frac{\partial f_0(\boldsymbol{k})}{\partial \boldsymbol{k}} = e(\boldsymbol{v_k} \times \boldsymbol{B}) \cdot \boldsymbol{v_k} \frac{\partial f_0(\varepsilon_{\boldsymbol{k}})}{\partial \varepsilon_{\boldsymbol{k}}} = 0. \tag{10}$$

Therefore, the magnetic field does not act as a source on the left hand side of Eq. (4) on the exercise sheet.

b) If we consider a magnetic field of the form $\boldsymbol{B} = (0, 0, B)$ and an electric field of the form $\boldsymbol{E} = (E, 0, 0)$, the linearized Boltzmann equation yields

$$-eEv_x \frac{\partial f_0}{\partial \epsilon} = -\frac{g(\boldsymbol{k})}{\tau} - \frac{eB}{\hbar}\left(v_{\boldsymbol{k}x}\frac{\partial g(\boldsymbol{k})}{\partial k_y} - v_{\boldsymbol{k}y}\frac{\partial g(\boldsymbol{k})}{\partial k_x}\right) . \tag{11}$$

If we now use the ansatz

$$g(\boldsymbol{k}) = -\frac{\partial f_0}{\partial \epsilon}(ak_x + bk_y) , \tag{12}$$

then the bracket on the right-hand-side in Eq. (11) is given by

$$\frac{1}{\hbar}\frac{\partial \epsilon_{\boldsymbol{k}}}{\partial k_x}\frac{\partial g(\boldsymbol{k})}{\partial k_y} - \frac{1}{\hbar}\frac{\partial \epsilon_{\boldsymbol{k}}}{\partial k_y}\frac{\partial g(\boldsymbol{k})}{\partial k_x} = -\frac{\partial f_0}{\partial \epsilon}\left(b\frac{\hbar k_x}{m} - a\frac{\hbar k_y}{m}\right) . \tag{13}$$

Eq. (11) then yields

$$k_x\left[\frac{e\hbar E}{m} + \frac{a}{\tau} + b\frac{eB}{m}\right] + k_y\left[\frac{b}{\tau} - a\frac{eB}{m}\right] = 0. \tag{14}$$

Since this has to be true for all values of k_x and k_y we find

$$a = -\frac{e\hbar E}{m}\frac{\tau}{1+\omega_c^2\tau^2} \qquad\qquad b = a\omega_c\tau = -\frac{e\hbar E}{m}\frac{\omega_c\tau^2}{1+\omega_c^2\tau^2}, \tag{15}$$

where $\omega_c = eB/m$. The coefficients a and b may now be used to calculate σ_{xx}, σ_{xy} and σ_{xz} with the last one obviously yielding zero; an example for σ_{xx} follows,

$$\begin{aligned} j_x &= \sigma_{xx}E_x = -2e\int \frac{d^3k}{(2\pi)^3} v_{\boldsymbol{k}x} g(\boldsymbol{k}) , & (16)\\ &= -E_x\frac{e^2(2m)^{3/2}}{4\pi^3 m\hbar^3}\int d\epsilon\, \epsilon^{3/2}\frac{\partial f_0}{\partial \epsilon}\frac{\tau}{1+\omega_c^2\tau^2}\underbrace{\int d\Omega \hat{k}_x(\hat{k}_x + \hat{k}_y\tau\omega_c)}_{4\pi/3} , & (17)\\ &= E_x\frac{e^2(2m\epsilon_F)^{3/2}}{3\pi^2\hbar^3 m}\frac{\tau}{1+\omega_c^2\tau^2} = E_x\frac{e^2 n}{m}\frac{\tau}{1+\omega_c^2\tau^2} . & (18)\end{aligned}$$

The other components can be found by repeating the calculation with E pointing in y- and z-direction. We find for the complete conductivity tensor

$$\hat{\sigma} = \frac{ne^2\tau}{m}\begin{pmatrix} \frac{1}{1+\alpha^2} & -\frac{\alpha}{1+\alpha^2} & 0 \\ \frac{\alpha}{1+\alpha^2} & \frac{1}{1+\alpha^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{19}$$

where $\alpha = \omega_c\tau$. The resistivity tensor is obtained by simply inverting $\hat{\sigma}$ and one finds

$$\hat{\rho} = \frac{m}{ne^2\tau}\begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{20}$$

We see immediately from Eq. (20) that the Hall resistance ρ_{xy} is independent of τ and that the transverse resistance is independent of the magnetic field and thus does not show a magnetoresistance.