Solid State Theory
Solution Sheet 10

Exercise 10.1 Polarization of a neutral Fermi liquid

We introduce the notation $\sigma^0 = \mathbf{1}$, $\sigma^1 = \sigma^x$, $\sigma^2 = \sigma^y$ and $\sigma^3 = \sigma^z$.

a) We write

$$\delta\hat{n}_{\vec{p}} = \sum_i \delta n^i_{\vec{p}}\sigma^i, \quad \hat{\epsilon}_{\vec{p}} = \sum_i \epsilon^i_{\vec{p}}\sigma^i \tag{1}$$

and

$$\hat{f}_{\vec{\sigma}\vec{\sigma}'}(\vec{p},\vec{p}') = \sum_{ij} \sigma^i f^{ij}(\vec{p},\vec{p}')\sigma^j \quad \text{where} \quad f^{ij} = \begin{cases} f^s, & i=j=0; \\ f^a, & i=j=1,2,3; \\ 0, & i \neq j. \end{cases} \tag{2}$$

The energy functional is then given by

$$\begin{aligned} E - E_0 &= \sum_{\vec{p},\alpha\beta} (\hat{\epsilon}_{\vec{p}})_{\alpha\beta}(\delta\hat{n}_{\vec{p}})_{\beta\alpha} + \frac{1}{2V}\sum_{\vec{p}\vec{p}'}\sum_{\alpha\beta\alpha'\beta'} (\delta\hat{n}_{\vec{p}})_{\beta\alpha}\hat{f}_{\alpha\beta,\alpha'\beta'}(\vec{p},\vec{p}')(\delta\hat{n}_{\vec{p}'})_{\beta'\alpha'} \\ &= \sum_{\vec{p}} \mathrm{tr}\left[\hat{\epsilon}_{\vec{p}}\,\delta\hat{n}_{\vec{p}}\right] + \frac{1}{2V}\sum_{\vec{p}\vec{p}',ij} \mathrm{tr}\left[\sigma^i\delta\hat{n}_{\vec{p}}\right] f^{ij}(\vec{p},\vec{p}')\,\mathrm{tr}\left[\sigma^j\delta\hat{n}_{\vec{p}'}\right] \\ &= 2\sum_{\vec{p},i}\epsilon^i_{\vec{p}}\,\delta n^i_{\vec{p}} + \frac{2}{V}\sum_{\vec{p}\vec{p}',i}\delta n^i_{\vec{p}}\,f^{ii}(\vec{p},\vec{p}')\,\delta n^i_{\vec{p}'}. \end{aligned} \tag{3}$$

In the last line we used the fact that $\mathrm{tr}[\sigma^i\sigma^j] = 2\delta_{ij}$.

b) For an electric field along the z direction the (bare) quasiparticle (QP) energy matrices have the form

$$\hat{\epsilon}_{\vec{p}} = \epsilon^0_{\vec{p}}\sigma^0 + \frac{\mu E_z}{2m^*c}\left(p_y\sigma^1 - p_x\sigma^2\right). \tag{4}$$

Thus, together with (3) the polarization in the z-direction is obtained as

$$P_z = \frac{\partial E}{\partial E_z} = \frac{\mu}{m^*c}\sum_{\vec{p}}\left(p_y\delta n^1_{\vec{p}} - p_x\delta n^2_{\vec{p}}\right). \tag{5}$$

c) We now consider the situation where the QP energies and the distribution of the quasiparticles are changed in linear response by the application of the electrical field. Thus, the *bare* QP energy changes according to

$$\hat{\epsilon}_{\vec{p}} = \hat{\epsilon}^0_{\vec{p}} + \delta\hat{\epsilon}_{\vec{p}} \tag{6}$$

where $\hat{\epsilon}^0_{\vec{p}} = \epsilon^0_{\vec{p}}\sigma^0$ is the bare quasiparticle energy in the absence of the electric field and

$$\delta\hat{\epsilon}_{\vec{p}} = \frac{\mu E_z}{2m^*c}\left(p_y\sigma^1 - p_x\sigma^2\right). \tag{7}$$

The components of the *dressed* QP energy are different from the bare values due to the QP interaction and will change according to

$$\delta\tilde{\epsilon}^i_{\vec{p}} = \delta\epsilon^i_{\vec{p}} + \frac{2}{V}\sum_{\vec{p}'} f^{ii}(\vec{p},\vec{p}')\delta n^i_{\vec{p}'}. \tag{8}$$

We can relate the change in the distribution function to the change in the QP energies in linear response

$$\delta n^i_{\vec{p}} = \frac{\partial n^0}{\partial \epsilon^0_{\vec{p}}} \delta \tilde{\epsilon}^i_{\vec{p}} = -\delta(\epsilon^0_{\vec{p}} - \epsilon_F) \delta \tilde{\epsilon}^i_{\vec{p}}. \tag{9}$$

The solution of the coupled Eqs. (8,9) can be found by using the ansatz

$$\delta \tilde{\epsilon}^i_{\vec{p}} = \alpha \delta \epsilon^i_{\vec{p}} \tag{10}$$

where α contains all the contributions from the QP interaction. Because

$$\delta \epsilon^i_{\vec{p}} = \frac{1}{2} \mathrm{tr} \left[\frac{\mu E_z}{2m^* c} \left(p_y \sigma^1 - p_x \sigma^2 \right) \sigma^i \right] = \frac{\mu E_z}{2m^* c} \times \begin{cases} p_y, & i = 1; \\ -p_x, & i = 2; \\ 0, & i = 0, 3; \end{cases} \tag{11}$$

it is clear that only the $i = 1, 2$ components will not vanish. Expanding the interaction parameters in terms of spherical harmonics [see Eq. (**??**)] as well as using the relations

$$\begin{aligned} p'_y &= \frac{1}{2i} \sqrt{\frac{8\pi}{3}} \left[Y_{11}(\theta', \phi') - Y_{1-1}(\theta', \phi') \right], & (12) \\ p'_x &= \frac{1}{2} \sqrt{\frac{8\pi}{3}} \left[Y_{11}(\theta', \phi') + Y_{1-1}(\theta', \phi') \right], & (13) \\ & & (14) \end{aligned}$$

it is straightforeward to show that

$$\begin{aligned} \frac{2}{V} \sum_{\vec{p}} f^a_{\vec{p}\vec{p}'} \delta n^{1,2}_{\vec{p}} &= -\alpha N(\epsilon_F) \int \frac{d\Omega'}{4\pi} f^a(\cos \theta_{\vec{p}\vec{p}'}) \delta \epsilon^{1,2}_{\vec{p}} & (15) \\ &= -\alpha N(\epsilon_F) \frac{f^a_1}{3} \times \begin{cases} p_y, & i = 1; \\ -p_x, & i = 2; \end{cases} & (16) \end{aligned}$$

where $N(\epsilon_F) = \frac{m^* k_F}{\pi^2 \hbar^2}$ is the density of states at the Fermi energy. Therefore,

$$\alpha = 1 - \alpha \frac{F^a_1}{3} \quad \Rightarrow \quad \alpha = \frac{1}{1 + \frac{F^a_1}{3}}. \tag{17}$$

d) Using the expression (5) we find for the polarization

$$\begin{aligned} P_z &= -\frac{\mu}{m^* c} \frac{N(\epsilon_F) V}{2} \alpha \int \frac{d\Omega}{4\pi} \left(p_y \delta \epsilon^1_{\vec{p}} - p_x \delta \epsilon^2_{\vec{p}} \right) & (18) \\ &= -\left(\frac{\mu}{m^* c} \right)^2 \frac{E_z}{2} \frac{N(\epsilon_F) V}{2} \alpha \int \frac{d\Omega}{4\pi} \underbrace{\left(p_y^2 + p_x^2 \right)}_{= \hbar^2 k_F^2 \sin^2 \theta} & (19) \\ &= -\left(\frac{\mu}{m^* c} \right)^2 \frac{E_z}{2} \frac{N(\epsilon_F) V}{2} \alpha \hbar^2 k_F^2 \underbrace{\frac{2\pi}{4\pi} \int_0^\pi d\theta \sin^3 \theta}_{= \frac{2}{3}} & (20) \\ &= -\frac{\mu^2}{2m^* c^2} \frac{N}{1 + \frac{F^a_1}{3}} E_z. & (21) \end{aligned}$$

Here we have used $\frac{N}{V} = \frac{k_F^3}{3\pi^2}$ where N is the total number of particles. Consequently, the susceptibility is given by

$$\chi = -\frac{\mu^2}{2m^*c^2}\frac{N}{1+\frac{F_1^a}{3}} \tag{22}$$

$$= -\frac{\epsilon_F}{m^*c^2}\frac{VN(\epsilon_F)}{3}\frac{\mu^2}{1+\frac{F_1^a}{3}}. \tag{23}$$

The response is dielectric.