ETH

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Solid State Theory
Solution Sheet 6

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Exercise 6.1 Lindhard function

At $T = 0$ the Fermi-Dirac distribution function $n_F(\epsilon_{\boldsymbol{k}})$ reduces to $\theta(\epsilon_F - \epsilon_{\boldsymbol{k}})$. As usual, we go from the discrete summation to a d-dimensional integral. Then, the static Lindhard function is given by

$$\chi_0(\boldsymbol{q}) \equiv \chi_0(\boldsymbol{q}, \omega = 0) = \frac{1}{\Omega} \sum_{\boldsymbol{k}} \frac{n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{q}}) - n_F(\epsilon_{\boldsymbol{k}})}{\epsilon_{\boldsymbol{k}+\boldsymbol{q}} - \epsilon_{\boldsymbol{k}} - i\hbar\eta} = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \frac{\theta(\epsilon_F - \epsilon_{\boldsymbol{k}+\boldsymbol{q}}) - \theta(\epsilon_F - \epsilon_{\boldsymbol{k}})}{\epsilon_{\boldsymbol{k}+\boldsymbol{q}} - \epsilon_{\boldsymbol{k}} - i\hbar\eta} \tag{1}$$

with the Fermi energy ϵ_F and the Heaviside step function

$$\theta(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}. \tag{2}$$

Next we split the integral and perform a change of variables in the second integral $(\boldsymbol{k} \to \boldsymbol{k} - \boldsymbol{q})$ such that

$$\chi_0(\boldsymbol{q}) = -\frac{1}{(2\pi)^d} \int \mathrm{d}^d k \; \theta(\epsilon_F - \epsilon_{\boldsymbol{k}}) \left(\frac{1}{\epsilon_{\boldsymbol{k}+\boldsymbol{q}} - \epsilon_{\boldsymbol{k}} - i\hbar\eta} - \frac{1}{\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}-\boldsymbol{q}} - i\hbar\eta} \right). \tag{3}$$

The dispersion relation for free electrons is given by $\epsilon_{\boldsymbol{k}} = \hbar^2 \boldsymbol{k}^2/2m$. We can therefore define the Fermi wave vector $k_F = \sqrt{2m\epsilon_F/\hbar}$ and the integration can be simplified further to

$$\chi_0(\boldsymbol{q}) = -\frac{1}{(2\pi)^d} \frac{2m}{\hbar^2} \int\limits_{|\boldsymbol{k}|<k_F} \mathrm{d}^d k \; \left(\frac{1}{\boldsymbol{q}(\boldsymbol{q} + 2\boldsymbol{k}) - i\hbar'\eta} + \frac{1}{\boldsymbol{q}(\boldsymbol{q} - 2\boldsymbol{k}) + i\hbar'\eta} \right). \tag{4}$$

where we introduced the abbreviation $\hbar' = 2m/\hbar$.

i) In the 1 dimensional case the integral is then simply

$$\chi_0(\boldsymbol{q}) = -\frac{1}{2\pi} \frac{2m}{\hbar^2} \int\limits_{-k_F}^{k_F} \mathrm{d}k \left(\frac{1}{q(q + 2k) - i\hbar'\eta} + \frac{1}{q(q - 2k) + i\hbar'\eta} \right). \tag{5}$$

We remark that for $|q| < 2k_F$ there is a singular point in the integral which is 'cured' by η meaning that in the limit $\eta \to 0$ we have to calculate the principle value since $\lim_{\eta\to 0}(z - i\eta)^{-1} = \mathcal{P}\,(1/z) + i\pi\delta(z)$.

For instance, let's consider the integral

$$\int\limits_{-k_F}^{k_F} \frac{\mathrm{d}k}{q + 2k} \tag{6}$$

for $|q| < 2k_F$. There is a singularity at $k = -q/2$ such that the integral is not well defined from a 'mathematical' point of view. However, the principle value

$$\mathcal{P} \int\limits_{-k_F}^{k_F} \frac{\mathrm{d}k}{q + 2k} = \lim_{\delta \to 0} \left(\int\limits_{-k_F}^{-q/2-\delta} \frac{\mathrm{d}k}{q + 2k} + \int\limits_{-q/2+\delta}^{k_F} \frac{\mathrm{d}k}{q + 2k} \right) \tag{7}$$

is well defined because there are no singularities within the integrals. We calculate then

$$\mathcal{P}\int_{-k_F}^{k_F} \frac{\mathrm{d}k}{q+2k} = \lim_{\delta\to 0}\left(\frac{1}{2}\log|q+2k|\Big|_{-k_F}^{-q/2-\delta} + \frac{1}{2}\log|q+2k|\Big|_{-q/2+\delta}^{k_F}\right) \tag{8}$$

$$= \lim_{\delta\to 0}\left(\frac{1}{2}\log\left|\frac{q+2k_F}{q-2k_F}\right| + \frac{1}{2}\log\left|\frac{-\delta}{\delta}\right|\right) = \frac{1}{2}\log\left|\frac{q+2k_F}{q-2k_F}\right|. \tag{9}$$

Therefore we can work with the anti-derivatives as if there are no singular points

$$\mathrm{Re}(\chi_0(\boldsymbol{q})) = -\frac{m}{\pi\hbar^2 q}\mathcal{P}\int_{-k_F}^{k_F}\mathrm{d}k\left(\frac{1}{q+2k}+\frac{1}{q-2k}\right) = -\frac{m}{\pi\hbar^2 q}\left(\frac{1}{2}\log\left|\frac{q+2k}{q-2k}\right|\right)\Bigg|_{-k_F}^{k_F} \tag{10}$$

$$= -\frac{m}{\pi\hbar^2 q}\log\left|\frac{q+2k_F}{q-2k_F}\right|. \tag{11}$$

ii) In the three dimensional case we assume $\boldsymbol{q} = q\,\boldsymbol{e}_z$ since the system is isotropic. The integral then reduces to

$$\chi_0(\boldsymbol{q}) = -\frac{1}{(2\pi)^3}\frac{2m}{\hbar^2}\int_{|\boldsymbol{k}|<k_F}\mathrm{d}^3k\left(\frac{1}{q(q+2k_z)-i\hbar'\eta}+\frac{1}{q(q-2k_z)+i\hbar'\eta}\right). \tag{12}$$

After a change to cylindrical coordinates ($k_x = r\cos(\phi), k_y = r\sin(\phi), k_z = k_z$ with $k^2 = r^2 + k_z^2 < k_F^2$) we get

$$-\frac{m}{4\pi^3\hbar^2}\int_{-k_F}^{k_F}\mathrm{d}k_z\int_0^{\sqrt{k_F^2-k_z^2}}\mathrm{d}r\ r\int_0^{2\pi}\mathrm{d}\phi\left(\frac{1}{q(q+2k_z)-i\hbar'\eta}+\frac{1}{q(q-2k_z)+i\hbar'\eta}\right). \tag{13}$$

The integration over r and ϕ are trivial and we find similarly as in (i) the real part of $\chi_0(\boldsymbol{q})$

$$\mathrm{Re}(\chi_0(\boldsymbol{q})) = -\frac{m}{2\pi^2\hbar^2 q}\mathcal{P}\int_{-k_F}^{k_F}\mathrm{d}k_z\frac{k_F^2-k_z^2}{2}\left(\frac{1}{q+2k_z}+\frac{1}{q-2k_z}\right). \tag{14}$$

Using partial fraction decomposition one can split the integrand into pieces which can then be integrated elementarily. After a straight-forward calculation we find

$$\int \mathrm{d}k_z\ k_z^2\left(\frac{1}{q+2k_z}+\frac{1}{q-2k_z}\right) = 2q\left[-\frac{k_z}{4}+\frac{q}{16}\log\left|\frac{q+2k_z}{q-2k_z}\right|\right] \tag{15}$$

and we finally obtain

$$\mathrm{Re}(\chi_0(\boldsymbol{q})) = -\frac{mk_F}{4\pi^2\hbar^2}\left[1-\frac{q}{4k_F}\left(1-\frac{4k_F^2}{q^2}\right)\log\left|\frac{q+2k_F}{q-2k_F}\right|\right]. \tag{16}$$

Note that in the both cases the imaginary part of $\chi_0(\boldsymbol{q})$ vanishes.

Exercise 6.2 Zero-sound excitations

a) Each infinitesimal volume element $\delta n(\boldsymbol{r},t)\mathrm{d}^3r$ contributes to the potential at $\boldsymbol{r'}$ with $V_{\text{local}}(\boldsymbol{r'}-\boldsymbol{r})$. Therefore the potential $V_{\text{ind}}(\boldsymbol{r'})$ which is induced by a particle distribution $\delta n(\boldsymbol{r},t)$ is

$$V_{\text{ind}}(\boldsymbol{r'},t) = \int \mathrm{d}^3r \; V_{\text{local}}(\boldsymbol{r'}-\boldsymbol{r})\delta n(\boldsymbol{r},t). \tag{17}$$

As we know, a convolution in real space corresponds to a simple multiplication in $(\boldsymbol{k},\omega)$-space and we get

$$V_{\text{ind}}(\boldsymbol{k},\omega) = V_{\text{local}}(\boldsymbol{k},\omega)\;\delta n(\boldsymbol{k},\omega) = U\;\delta n(\boldsymbol{k},\omega) \tag{18}$$

where we used that the Fourier transform of the δ-function is 1.

b) As in the lecture notes we apply an external potential $V_a(\boldsymbol{q},\omega)$ to the system and calculate the resulting potential $V(\boldsymbol{q},\omega)$ which consists of both, the external potential V_a and the induced potential V_{ind}. With the induced particle distribution

$$\delta n(\boldsymbol{q},\omega) = \chi_0(\boldsymbol{q},\omega)\;V(\boldsymbol{q},\omega) \tag{19}$$

and the potential induced by δn

$$V_{\text{ind}} = U\delta n(\boldsymbol{q},\omega) \tag{20}$$

we find with $V = V_{\text{ind}} + V_a$

$$V(\boldsymbol{q},\omega) = \frac{V_a(\boldsymbol{q},\omega)}{1-U\chi_0(\boldsymbol{q},\omega)} \equiv \frac{V_a(\boldsymbol{q},\omega)}{\epsilon(\boldsymbol{q},\omega)} \tag{21}$$

where $\epsilon(\boldsymbol{q},\omega) = 1 - U\chi_0(\boldsymbol{q},\omega)$ is the dynamical dielectric function. The response function $\chi(\boldsymbol{q},\omega)$ is therefore

$$\chi(\boldsymbol{q},\omega) = \frac{\chi_0(\boldsymbol{q},\omega)}{\epsilon(\boldsymbol{q},\omega)} = \frac{\chi_0(\boldsymbol{q},\omega)}{1-U\chi_0(\boldsymbol{q},\omega)}. \tag{22}$$

Next we expand the dispersion relation $\epsilon_{\boldsymbol{k}+\boldsymbol{q}}$ about $\boldsymbol{q}=0$ and calculate $\chi_0(\boldsymbol{q},\omega)$ in lowest order of $|\boldsymbol{q}|$ (see Eq. (3.66) in section 3.2.2 for the details) which yields

$$\chi_0(\boldsymbol{q},\omega) \approx \frac{k_F^3 q^2}{3\pi^2 m(\omega+i\eta)^2} = \alpha^2\frac{q^2}{(\omega+i\eta)^2} \tag{23}$$

with the abbreviation $\alpha = \sqrt{k_F^3/(3\pi^2 m)}$ and $q = |\boldsymbol{q}|$. Then we put this expression in Eq. (22) and obtain

$$\chi(\boldsymbol{q},\omega) = \frac{\alpha^2 q^2}{(\omega+i\eta)^2 - U\alpha^2 q^2} = \frac{\alpha q}{2\sqrt{U}}\left(\frac{1}{\omega+i\eta-\sqrt{U}\alpha q} - \frac{1}{\omega+i\eta+\sqrt{U}\alpha q}\right). \tag{24}$$

Splitting the response function into real and imaginary part by using the identity $\lim_{\eta\to 0}(z-i\eta)^{-1} = \mathcal{P}\,(1/z) + i\pi\delta(z)$ we find that the imaginary part is given by

$$\operatorname{Im}(\chi(\boldsymbol{q},\omega)) \sim \left(\delta(\omega-\sqrt{U}\alpha q) - \delta(\omega+\sqrt{U}\alpha q)\right). \tag{25}$$

The excitation modes are therefore

$$\omega_{\boldsymbol{q}} = \sqrt{U}\alpha|\boldsymbol{q}| = \sqrt{\frac{k_F^3 U}{3\pi^2 m}}|\boldsymbol{q}|. \tag{26}$$

In contrast to the Coulomb potential, the local potential leads to plasmon excitations with a linear dispersion relation which vanishes at $\boldsymbol{q}=0$.

c) If the dispersion relation of plasmons lies in the region of particle-hole excitations, it will be damped and the plasmon excitation will have only a short lifetime. Thus, it is favourable for the plasmons to have a dispersion relation which is outside of the particle-hole continuum. Since both, the dispersion relation of plasmons and the upper boundary line for particle-hole excitations, goes linearly to zero as $\boldsymbol{q} \to 0$ the plasmon excitation is stable if the slope of the plasmon dispersion is larger then the slope of the boundary line. Therefore we find the condition

$$\sqrt{\frac{k_F^3 U}{3\pi^2 m}} > v_F = \frac{\hbar k_F}{m} \tag{27}$$

and finally

$$U > U_c = \frac{3\pi^2 \hbar^2}{m k_F}. \tag{28}$$