ETH
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Solid State Theory
Solution Sheet 13

FS 12
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Exercise 13.1 Critical temperature in the Stoner model

In the lecture, it was shown that the critical temperatures obeys the equation

$$k_B T_c = \frac{\sqrt{6}}{\pi \Lambda_1(\epsilon_F)} \sqrt{1 - \frac{2}{UN(\epsilon_F)}}, \tag{1}$$

with

$$\Lambda_1(\epsilon_F)^2 = \Big(\frac{N'(\epsilon_F)}{N(\epsilon_F)}\Big)^2 - \frac{N''(\epsilon_F)}{N(\epsilon_F)}. \tag{2}$$

We therefore first need to calculate the densities of states and then use eq. (1) to calculate T_c for different chemical potentials μ.

- $\epsilon_{\boldsymbol{k}} = \epsilon_0 \pm \frac{\hbar^2 \boldsymbol{k}^2}{2m}$: The density of states is given by

$$N(\epsilon) = \frac{(2m)^{3/2}}{\hbar^3 \pi^2} \sqrt{\mp \epsilon_0 \pm \epsilon}. \tag{3}$$

  Therefore, $\Lambda_1(\epsilon_F)$ is given by

$$\Lambda_1(\epsilon_F)^2 = \Big(\frac{N'(\epsilon_F)}{N(\epsilon_F)}\Big)^2 - \frac{N''(\epsilon_F)}{N(\epsilon_F)} = \frac{1}{2(\mp\epsilon_0 \pm \epsilon_F)^2}. \tag{4}$$

  The equation for the critical temperature now reads

$$k_B T_c = \frac{2\sqrt{3}}{\pi} |\epsilon_0 - \epsilon_F| \sqrt{1 - \frac{2\pi^2 \hbar^3}{(2m)^{3/2} U \sqrt{|\epsilon_0 - \epsilon_F|}}}. \tag{5}$$

  Introducing the energy scale

$$E_U = \frac{\pi^4 \hbar^6}{((2m)^3 U^2}, \tag{6}$$

  note that U has dimension (energy $\times$ length^3), we can express $N(\epsilon_F)U = \sqrt{|\epsilon_F - \epsilon_0|/E_U}$ and

$$\frac{k_B T_c}{E_U} = \frac{2\sqrt{3}}{\pi} (|\epsilon_F - \epsilon_0|/E_U) \sqrt{1 - \frac{2}{\sqrt{|\epsilon_F - \epsilon_0|/E_U}}} \tag{7}$$

  which is plotted in Fig. 1.

- For the one dimensional case with linear dispersion we simply get a constant density of states. Therefore, Λ_1 vanishes and according to eq. (1) we would expect an infinite T_c, at least if the Stoner criterion is fulfilled.
  However, since we divided by zero, we should be more careful. Looking at the derivation of eq. (1) we see that we actually started with an expression for the magnetization which read (cf. eq. (7.15) of the script)

$$m = -\frac{1}{2} \sum_s \int d\epsilon \, s \, N(\epsilon - \frac{Un_0}{2} - s\frac{Um}{2}) f(\epsilon) \tag{8}$$

  which for a constant density of states just vanishes. We therefore have no ferromagnetism at all and thus no critical temperature T_c.

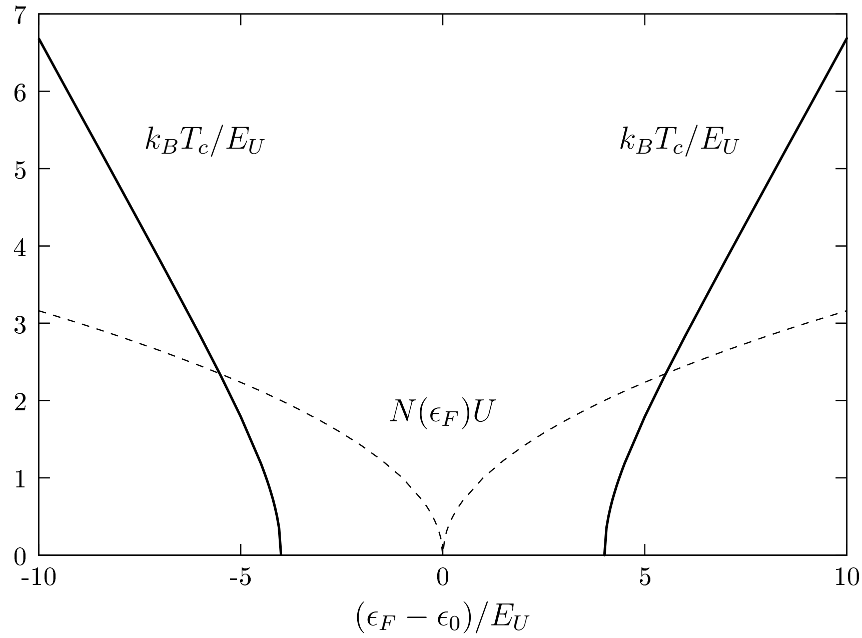

Figure 1: Critical Temperature and the density of states (dashed line) as a function of the Fermi energy. Note that the magnetization becomes non-zero (the critical temperature becomes finite) when $N(\epsilon_F)U > 2$.

Exercise 13.2 Stoner instability

The total energy of the system at $T = 0$ is given as

$$E_{\text{tot}} = \langle \Psi_G | \mathcal{H}_{\text{MF}} | \Psi_G \rangle \tag{9}$$

with $|\Psi_G\rangle$ the ground state of the system, given by a filled Fermi sea up to energies $\epsilon_{F\uparrow(\downarrow)}$ for electrons with spin up (down).
We can thus write the energy as

$$\begin{aligned}
E_{\text{tot}} &= \frac{1}{\Omega}\sum_s \sum_{|\boldsymbol{k}|<k_{Fs}} (\epsilon_{\boldsymbol{k}} + Un_{-s}) - Un_\uparrow n_\downarrow \\
&= \frac{1}{(2\pi)^3}\sum_s \int_{|\boldsymbol{k}|<k_{Fs}} d^3k(\epsilon_{\boldsymbol{k}} + Un_{-s}) - Un_\uparrow n_\downarrow \\
&= \left[\left(\int_0^{k_{F_\uparrow}} + \int_0^{k_{F_\downarrow}}\right) \frac{4\pi k^2 dk}{(2\pi)^3}\frac{\hbar^2 k^2}{2m}\right] + Un_\uparrow n_\downarrow \qquad (10)\\
&= \frac{\hbar^2}{2m} 2^{2/3}(3\pi^2)^{2/3}\frac{3}{5}(n_\uparrow^{5/3} + n_\downarrow^{5/3}) + Un_\uparrow n_\downarrow \qquad (11)
\end{aligned}$$

where $n_{\uparrow(\downarrow)}$ is the density of electrons with up (down) spin (here, we have made use of the following expression, i.e. $n_\uparrow = k_{F_\uparrow}^3/(6\pi^2)$ and $n_\downarrow = k_{F_\downarrow}^3/(6\pi^2)$). We now express these densities with the relative number of electrons with up (down) spin,

$$n_{\uparrow(\downarrow)} = \frac{N_{\uparrow(\downarrow)}}{N_e}\frac{N_e}{N} = \frac{1}{2}(1 \pm x)n. \tag{12}$$

Here, N is the total number of sites, N_e is the total number of electrons and $N_{\uparrow(\downarrow)}$ is the total number of electrons with up (down) spin.

Introducing this into eq. (11) we find

$$E_{\text{tot}} = \frac{\hbar^2}{4m}(3\pi^2)^{2/3}\frac{3}{5}((1+x)^{5/3} + (1-x)^{5/3})n^{5/3} + \frac{U}{4}(1-x^2)n^2. \tag{13}$$

The condition for a minimum of the total energy is found by differentiating eq. (13) with respect to x,

$$2Ux = \frac{\hbar^2(3\pi^2)^{2/3}}{m\,n^{1/3}}((1+x)^{2/3} - (1-x)^{2/3}) \tag{14}$$

We can now use the expression for the density of states, $N(\epsilon_F) = 3n^{1/3}m/(\hbar^2(3\pi^2)^{2/3})$, to find

$$UN(\epsilon_F) = \frac{3}{2x}((1+x)^{2/3} - (1-x)^{2/3}). \tag{15}$$

This equation has only non-vanishing solutions for $UN(\epsilon_F) > 2$ which can be seen by plotting both the right hand side of Eq. (15) (see Fig. 2(a)) and we can find the solutions for x numerical, see Fig. 2(b). For $N(\epsilon_F)U > 3/2^{1/3}$, the system is completely polarized and $x = \pm 1$. For $N(\epsilon_F)U < 2$, the system is unpolarized and $x = 0$ (this solution is absent in Eq. (15) as we divided by x to obtain this equation).

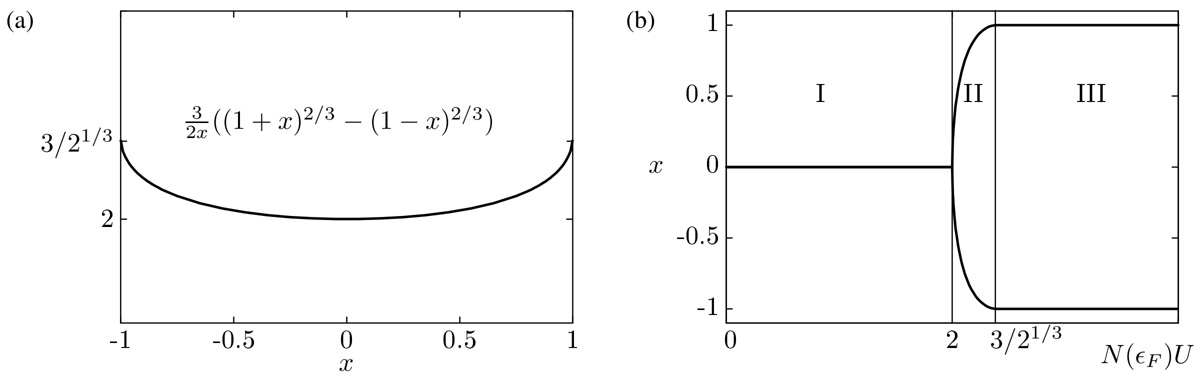

Figure 2: Graphical solution of Eq. (15): (a) The right hand side is larger than 2, therefore the first non-vanishing solution appears as soon as $UN(\epsilon_F) > 2$. (b) As long as $UN(\epsilon_F) < 2$, the system is in a paramagnetic state (I). With increasing $UN(\epsilon_F)$, the polarization increases as well (weakly polarized ferromagnetic state (II), until it reaches its maximum value of ± 1 at $UN(\epsilon_F) = 3/2^{1/3}$ and we have a completely polarized ferromagnetic state (III).