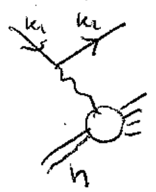
- Introduction

In the first part of the course we have learned how to perform calculations for processes involving electromagnetic interactions in QED. Similar techniques can be applied in QCD to describe the interactions of quarks and gluons in QCD.

But quarks and gluons are not free objects: they are bound into composite particles called hadrons

⇒ in these first lectures we will study a technique that allows us to determine the hadron structure in terms of quarks and gluons (Deep inelastic scattering)



- Studying an object by bombarding it with a beam of charged particles is a well known technique in physics

- Rutherford scattering

alpha particles against gold nuclei ⇒ small fraction of particles deflected at very large angles

⇒ impossible to explain with plum pudding model of atom by Thompson (electrons distributed in a sea of positive charges)

large angles imply that positive charge is concentrated in the nucleus

We consider the scattering of an electron (non relativistic spinless) off a nucleus of charge Ze

plane wave wavefunctions in the box

$$\phi_{\underline{k}}(\underline{x},t) = \sqrt{C}\, e^{-i(Et - \underline{k}\cdot\underline{x})} \qquad E = \frac{|\underline{k}|^2}{2m}$$

arbitrary constant

$\underline{k} = \frac{2\pi}{L}\underline{n}$ $\underline{n} = (n_1, n_2, n_3)$ $N_V = \int_V d^3\underline{x}\, \phi_n^* \phi_n = cV$

$$T_{fi} = -\int d^4x\, c\, e^{i(E_f t - \underline{k}_f\cdot\underline{x})}\, V(\underline{x})\, e^{-i(E_i t - \underline{k}_i\cdot\underline{x})} = -2\pi\,\delta(E_f - E_i)\, c\, V(\underline{q})$$

$$V(\underline{q}) = \int d^3x\, e^{+i\underline{q}\cdot\underline{x}}\, V(\underline{x})$$ Fourier Transform of the potential $\underline{q} = \underline{k}_f - \underline{k}_i$

$\underline{k}_f = \frac{2\pi}{L}\underline{n}$ $\Rightarrow$ number of final states is $d^3\underline{n} = \frac{V}{(2\pi)^3} d^3\underline{k}$

number of final states for each particle is $\frac{V}{(2\pi)^3}\frac{d^3\underline{k}}{N_V} = \frac{d^3\underline{k}}{(2\pi)^3 c}$

The transition probability for unit of time is $w = \frac{|T_{fi}|^2}{T}$

$$|2\pi\,\delta(E_f - E_i)|^2 \longrightarrow 2\pi T\,\delta(E_f - E_i) \quad \left(|2\pi\,\delta(E_f-E_i)|^2 = 2\pi\,\delta(E_f-E_i)\underbrace{\int_{-T/2}^{T/2} e^{i(E_f-E_i)t}\,dt}_{\to T} \right)$$

The number of scattered particles per unit of time between $\underline{k}_f$ and $\underline{k}_f + d\underline{k}_f$ is

$$dN = w\,\frac{d^3\underline{k}_f}{(2\pi)^3 c}$$

To construct the cross section we need the number of particles per unit of time and surface

$\Rightarrow$ it is the modulus of the probability current

$$\mathcal{M} = |\underline{j}| = \left| -\frac{i}{2m}\left(\phi_i^* \nabla\phi_i - (\nabla\phi)\phi_i\right)\right| = c\,|\underline{v}_i|$$

$$\Rightarrow\ d\sigma = \frac{dN}{\mathcal{M}} = w\,\frac{1}{c|\underline{v}_i|}\,\frac{d^3\underline{k}_f}{(2\pi)^3 c} = 2\pi\,\delta(E_f - E_i)\,\cancel{c^2}\,|V(\underline{q})|^2\,\frac{1}{\cancel{c^2}|\underline{v}_i|}\,\frac{d^3\underline{k}_f}{(2\pi)^3}$$

$$d^3\underline{k} = |\underline{k}|^2\, d|\underline{k}|\, d\Omega \qquad \left(E = \frac{\underline{k}^2}{2m} \quad dE = \frac{|\underline{k}|}{m}\, d|\underline{k}| \right)$$

$$= 2mE\,\frac{m}{|\underline{k}|}\, dE\, d\Omega$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2} |V(\underline{q})|^2$$

the cross section is proportional to the square of the Fourier transform of the potential

Let us specialise to the Coulomb potential and compute its Fourier Transform

$$-e\,\phi(x) = V(x) = \int \frac{d^3q}{(2\pi)^3} e^{-i\underline{q}\cdot\underline{x}} V(\underline{q})$$

$$\nabla^2\phi = -\rho(x) = -Ze\,\delta^3(x)$$ simple positive charge Ze

$$e\rho(x) = e^2 Z\,\delta^3(x) = -\int \frac{d^3q}{(2\pi)^3} e^{i\underline{q}\cdot\underline{x}} |\underline{q}|^2 V(\underline{q}) \quad \Rightarrow \quad V(\underline{q}) = -\frac{Ze^2}{\underline{q}^2}$$

$$q^2 = |k_f - k_i|^2 = 2\underline{k}^2 - 2\underline{k}^2\cos\theta = 2k^2(1-\cos\theta) = 4k^2\sin^2\theta/2$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2}\frac{Z^2e^4}{16\underline{k}^4\sin^4\theta/2} = \boxed{\frac{Z^2\alpha^2}{16E^2\sin^4\theta/2}}$$ nr

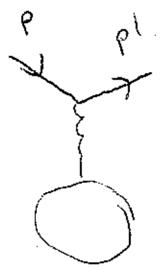
- Mott scattering

We now consider the scattering of a relativistic electron (with spin) off a distributed Coulomb potential

$$A^\mu = (\phi, \underline{0})$$

$$\phi(\underline{r}) = \int \frac{\rho(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} d^3r'$$

$$\int \rho(r')d^3r' = Ze$$



The scattering matrix is

$$T_{fi} = -i\int J^\mu(x) A_\mu(x)\, d^4x$$

$$J_\mu = \bar{\Psi}(x)\gamma_\mu\Psi(x)\, e$$

④

$$T_{fi} = -ie \int \bar{u}(p') e^{ip'x} \not{A}\, u(p) e^{-ipx} d^4x$$

$$= -ie\, \bar{u}(p') \gamma_0 u(p) \int dt\, d^3x\, e^{-i(E-E')t} e^{i(\underline{p}'-\underline{p})\cdot \underline{x}} A_0(\underline{x})$$

$$= -ie\, 2\pi \delta(E-E')\, \bar{u}(p')\gamma_0 u(p) \underbrace{\int e^{-i\underline{q}\cdot\underline{x}} d^3x\, A_0(\underline{x})}_{= A_0(\underline{q})}$$

Let us square and sum over spins

$$\frac{1}{2}\sum_{s,s'} |\bar{u}(p')\gamma^0 u(p)|^2 = \frac{1}{2}\sum_{s,s'} \left(\bar{u}(p')\gamma^0 u(p)\bar{u}(p)\gamma^0 u(p')\right)$$

$$= \frac{1}{2} \mathrm{Tr}\left[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right] = \frac{1}{2}\left[\mathrm{Tr}(\not{p}'\gamma^0\not{p}\gamma^0) + m^2 \mathrm{Tr}(\gamma^0\gamma^0)\right]$$

$$= \frac{1}{2}4\left(2EE' - pp' + m^2\right) = 2\left(2EE' - (EE' - pp'\cos\theta) + m^2\right)$$

exploit $E'=E$

$$= 2\left(E^2 + \underline{p}^2\cos\theta + E^2 - \underline{p}^2\right) = 2\left(2E^2 - \underline{p}^2(1-\cos\theta)\right)$$

$$= 4\left(E^2 - \underline{p}^2 \sin^2\theta/2\right) = 4E^2\left(1 - v^2\sin^2\theta/2\right)$$

$$w = \frac{|T_{fi}|^2}{T} = e^2\, 2\pi\, \delta(E-E')\, 4E^2(1-v^2\sin^2\theta/2)\, |A_0(\underline{q})|^2$$

$$d\sigma = \frac{w}{I}\frac{d^3p'}{(2\pi)^3 2E'} \qquad I = 2Ev \quad \text{flux factor}$$

$E'^2 - \underline{p}'^2 = m^2$

$\Rightarrow E'dE' = p'dp'$

$$= 2\pi e^2 \delta(E-E')\, 4E^2(1-v^2\sin^2\theta/2)\, |A_0(\underline{q})|^2 \frac{1}{2Ev}\frac{p'^2 dp'\, d\Omega}{(2\pi)^3 2E'}$$

$$= 2\pi e^2\, 4E^2 (1-v^2\sin^2\theta/2)\, |A_0(\underline{q})|^2 \frac{1}{2\cancel{E}\cancel{v}} \frac{\cancel{p'}\cancel{E}\, d\Omega}{(2\pi)^3 2\cancel{E}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} e^2\, E^2 (1-v^2\sin^2\theta/2)\, |A_0(\underline{q})|^2$$

$$A_0(\underline{q}) = \int e^{-i\underline{q}\cdot\underline{x}}\, d^3x\, A_0(\underline{x}) = \int d^3x\, e^{-i\underline{q}\cdot\underline{x}} \frac{\rho(x')}{4\pi|\underline{x}-\underline{x}'|}\, d^3x'$$

$$= \int \rho(x')\, d^3x' \int d^3x \frac{e^{-i\underline{q}\cdot\underline{x}}}{4\pi|\underline{x}-\underline{x}'|} = \int \rho(x')\, d^3x'\, e^{-i\underline{q}\cdot\underline{x}'} \int d^3\tilde{x} \frac{e^{-i\underline{q}\cdot\tilde{x}}}{4\pi|\tilde{x}|}$$

$\tilde{x} = x - x'$

$$= -\int \rho(x')\, d^3x'\, e^{-i\underline{q}\cdot\underline{x}'} \frac{1}{\underline{q}^2}$$

use result from Rutherford scattering!

$$\equiv -\frac{1}{\underline{q}^2} F(\underline{q})\, Ze$$

$F(\underline{q})$ FORM FACTOR (note that since $\int \rho = Ze$ $\Rightarrow F(0) = 1$)

$$\underline{q}^2 = (\underline{p}'-\underline{p})^2 = 2\underline{p}^2 - 2\underline{p}^2\cos\theta = 4\underline{p}^2 \sin^2\theta/2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\cancel{4}E^2(1-v^2\sin^2\theta/2)}{\cancel{16}_4\, \underline{p}^4 \sin^4\theta/2} |F(q)|^2 Z^2\alpha^2 = \boxed{\frac{(Z\alpha)^2 E^2(1-v^2\sin^2\theta/2)}{4\underline{p}^4 \sin^4\theta/2} |F(q)|^2}$$

Mott cross section

COMMENTS

- When $v \ll 1$ the behavior is the same seen in Rutherford scattering (scalar electrons) The reason is that for a non relativistic electron what matters is the electric interaction, that cannot change the spin. At high energies the interaction of the spin with the magnetic field becomes relevant

- If $\underline{q}$ is not large

$$F(\underline{q}) = \frac{1}{Ze} \int \rho(x)\, d^3x\, e^{-i\underline{q}\cdot\underline{x}} \simeq \frac{1}{Ze} \int \rho(x)\, d^3x \left(1 - i\underline{q}\cdot\underline{x} - \frac{1}{2}(\underline{q}\cdot\underline{x})^2 + \ldots\right)$$

if ρ is spherically symmetric the linear term does not contribute and we get

$$\frac{1}{\cancel{Ze}} \cancel{Ze} \left(1 - \frac{1}{6} |\underline{q}|^2 \langle r^2\rangle + \ldots\right) \quad (*)$$

SMALL ANGLE SCATTERING JUST MEASURE THE MEAN SQUARE RADIUS OF THE CHARGE CLOUD (SOFT PHOTON CANNOT RESOLVE THE DETAILS OF THE CHARGE DISTRIBUTION)

(*) In 1967 SLAC started to operate electrons up to a maximum energy of about 22 GeV and to collide them on a fixed target of protons

We can qualitatively describe the results as follows

$ep \to ep$ elastic scattering with $|\underline{q}|$ up to about 3 GeV

$$F(\underline{q}) = \left(1 + \frac{|\underline{q}|^2}{0.71}\right)^{-2}$$

experimentally observed $|\underline{q}|^2$ dependence (where $|\underline{q}|$ is in GeV)

$$\Rightarrow \quad \langle |\underline{r}|^2 \rangle = -6 \left. \frac{dF}{d|\underline{q}|^2} \right|_{\underline{q}=0} = 16.9 \ \mathrm{GeV}^{-2} \sim (0.8 \cdot 10^{-13}\ \mathrm{cm})^2$$

Electron - proton scattering : the elastic case

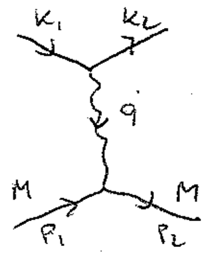
Can we apply directly what we have seen so far to study the proton structure?

The answer is no for two reasons.

- the proton is not simply a static charge, but it also has a magnetic moment
- the proton will recoil, and we have to take this into account

Let us suppose that the proton is pointlike, and has a mass M (replace it with a muon)

We can compute the scattering cross section by evaluating the diagram below (neglect electron mass)



$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2 \sin^4\frac{\vartheta}{2}} \frac{E'}{E} \left(\cos^2\frac{\vartheta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\vartheta}{2} \right)$$

↑ scattering off a pointlike heavy fermion

(Note that in the limit $M \to \infty$ we recover the $v \to 1$ limit of the Mott cross section)

The result arises from the assumption that the scattering matrix has the form

$$T_{fi} = -i \int \underset{\text{leptonic}}{j_\mu} \left(-\frac{1}{q^2} \right) \underset{\text{hadronic current}}{J^\mu}$$

($q^\mu q^\nu$ terms in the propagator vanish by gauge invariance)

$$j_\mu = -e\, \bar{u}(k_2)\gamma_\mu u(k_1)\, e^{i(k_2-k_1)x}$$

$$J_\mu = e\, \bar{u}(P_2) \left[\ ? \ \right] u(P_1)\, e^{i(P_2-P_1)x}$$

what is the general structure of the current?

Most general structure is

$$\bar{u}(P_2)\left[F_1\gamma_\mu + G_1 \sigma_{\mu\nu}P_2^\nu + G_2\sigma_{\mu\nu}P_1^\nu + F_3 (P_1+P_2)_\mu + F_4 q_\mu \right] u(P_1)$$

where F_i and G_i are scalar functions of the only Lorentz scalar that can be constructed out of P_1 and P_2, that is q^2 ($q = k_1 - k_2 = P_2 - P_1$)

Using current conservation we get (we can use the Dirac equation since we have a proton of $\frac{1}{2}$ spin)

$$q^\mu \bar{u}(P_2)[\ \ ] u(P_1) = \bar{u}(P_2)\Big[F_1(\not{P}_2 - \not{P}_1) - G_1 \sigma_{\mu\nu}P_2^\nu P_1^\mu + G_2\sigma_{\mu\nu}P_1^\nu P_2^\mu$$
$$+ F_3(P_2^2 - P_1^2) + F_4 q^2 \Big] u(P_1)$$

$$= \bar{u}(P_2)\left[-G_1\sigma_{\mu\nu}P_1^{\mu}P_2^{\nu} + G_2\sigma_{\mu\nu}P_1^{\nu}P_2^{\mu} + F_4 q^2\right]u(P_1)$$

$$\Rightarrow \quad G_1 = -G_2 \qquad F_4 = 0$$

To eliminate F_3 we can use the Gordon identity (*)

$$\bar{u}(P_2)\gamma_\mu u(P_1) = \frac{1}{2m}\bar{u}(P_2)\left[(P_1+P_2)_\mu + i\sigma_{\mu\nu}(P_2-P_1)^\nu\right]u(P_1)$$

that allows us to finally have only two possible functions

$$\boxed{[\ \ ] = F_1(q^2)\gamma_\mu + \frac{\kappa}{2M}F_2(q^2)\, i\,\sigma_{\mu\nu}q^\nu}$$

$\kappa = 1.79$
anomalous
magnetic moment

In the limit $q^2 \to 0$ we don't see at all the proton structure

we effectively see a positive charge e and a magnetic moment $\frac{(1+\kappa)e}{2M}$

$$\Rightarrow \quad F_1(0) = 1 \quad \text{and} \quad F_2(0) = 1$$

(*) Note that Gordon identity allows us to separate the standard electromagnetic interaction in terms of an electric (charge) and magnetic component

$$\bar{u}(P_2)\gamma_\mu u(P_1) = \frac{1}{2m}\bar{u}(P_2)\left[(P_1+P_2)_\mu + i\sigma_{\mu\nu}(P_2-P_1)^\nu\right]u(P_1)$$

scalar coupling
(charge)

magnetic interaction

$$\int\left[-\frac{e}{2m}\bar{\Psi}\, i\,\sigma_{\mu\nu}(P_2-P_1)^\nu\Psi\right]A^\mu\, d^3x = \int \Psi_A^{+}\left(\frac{e}{2m}\,\underline{\sigma}\cdot\underline{B}\right)\Psi_A\, d^3x$$

↳ upper component in the non relativistic limit

$$\underline{\mu} = -g\frac{e}{2m}\underline{s}$$

g gyromagnetic ratio

$g \simeq 2$ for the electron

By using the expression of the hadronic current in terms of the form factors F_1 and F_2 we can compute the differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left\{ \left(F_1^2 - \frac{k^2 q^2}{4M^2} F_2^2 \right) \cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \left(F_1 + k F_2 \right)^2 \sin^2\frac{\theta}{2} \right\}$$

ROSENBLUTH FORMULA

$$G_E = F_1 + \frac{k q^2}{4M^2} F_2$$

$\Rightarrow$ avoids interference term

$$G_M = F_1 + k F_2$$

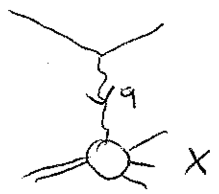
electric and magnetic form factors $\Rightarrow$ Fourier transform of the charge and magnetic moment of the nucleon?

No because of nucleon recoil!

Electron proton scattering : the inelastic case

Having measured the size of the proton, we may want to have a closer look into it by increasing the q^2 of the virtual photon to have a better resolution. However in this case the proton will break up. For not too large q^2 we typically excite a Δ^+ that then produces $p\pi^0$

$ep \to e\Delta^+ \to ep\pi^0$. However when q^2 becomes larger the final state becomes more and more complicated, and the proton looses its identity.



However the interaction of the proton with the virtual photon can still be described through the electromagnetic current evaluated between the proton and the final state X

$$\langle X | J_\mu(0) | P \rangle$$

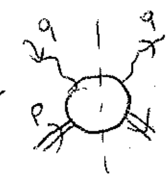
Such that the cross section is

$$d\sigma \sim L_{\mu\nu} W^{\mu\nu}$$

$L_{\mu\nu}$ obtained by squaring the leptonic part $\bar{u}(k_2)\gamma_\mu u(k_1)$

and $W_{\mu\nu} \propto \langle P | J_\mu(0) | X \rangle \langle X | J_\nu(0) | P \rangle$

If we are <u>inclusive</u> (that is, if we sum over X) we can define

$W_{\mu\nu} \sim$ 

$$W_{\mu\nu} = \frac{1}{4\pi M} \sum_X (2\pi)^4 \delta^4(P_X - P - q) \langle P | J_\mu(0) | X \rangle \langle X | J_\nu(0) | P \rangle$$

overall arbitrary normalization

$$= \frac{1}{4\pi M} \int d^4y\, e^{iqy} \langle P | J_\mu(y) J_\nu(0) | P \rangle$$

where we have used completeness and translational invariance

$$\left(2\pi\, \delta^4(P_X - P - q) = \int d^4y\, e^{-i(P_X - P - q)y} \right)$$

What is the most general structure of $W_{\mu\nu}$?

We can limit ourselves to consider SYMMETRIC tensors, since the leptonic tensor is symmetric

We can put all the second rank tensors that can be constructed out of P_μ and q_μ,

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} P^\mu P^\nu + \frac{W_4}{M^2} q^\mu q^\nu + \frac{W_5}{M^2} (P^\mu q^\nu + P^\nu q^\mu) \quad *$$

$$W^{\mu\nu} q_\nu = -W_1 q^\mu + \frac{W_2}{M^2} P^\mu (Pq) + \frac{W_4}{M^2} q^2 q^\mu + \frac{W_5}{M^2} (P^\mu q^2 + q^\mu Pq) = 0$$

current conservation

setting to zero the coefficient of P^μ we get $W_5 = -W_2 \frac{Pq}{q^2}$

* If parity is violated we would have also another function W_3

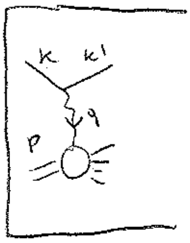
Replacing W_5 in terms of W_2 we get

$$-W_1 q^\mu + \frac{W_2}{M^2}\cancel{p^\mu (pq)} + \frac{W_4}{M^2} q^2 q^\mu + (\cancel{p^\mu q^2} + q^\mu pq)\left(-\frac{W_2}{M^2}\right)\frac{pq}{q^2}$$

$$= -W_1 q^\mu + \frac{W_4}{M^2} q^2 q^\mu - \frac{W_2}{M^2} q^\mu \frac{pq^2}{q^2} \qquad \Rightarrow \quad \frac{W_4}{M^2} q^2 = W_1 + \frac{W_2}{M^2}\frac{(pq)^2}{q^2}$$

$$\Rightarrow W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{M^2}\left(p^\mu p^\nu + \dots\right)$$

$$= W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{M^2}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right) \qquad [W_i] = m^{-1}$$



The functions W_i depend on the Lorentz scalars that we have at our disposal

Contrary to the elastic case we have two independent scalars: q^2 and $\nu = \frac{pq}{M}$

We often use $\quad x = -\frac{q^2}{2pq} \qquad y = \frac{pq}{pk}$

Neglect hadron masses ⇒ DIS limit imply $-q^2$, $2pq$ large and x finite

$q = (k-k')^2 = -2kk' < 0$

$(p+q)^2 \simeq q^2 + 2pq \geq 0 \quad \Rightarrow \boxed{0 \leq x \leq 1}$

(valid also for $p^2 = M^2 \neq 0$ $\quad (p+q)^2 \geq M^2 \Rightarrow x \leq 1$)

$y = \frac{E-E'}{E}$ in rest frame of the proton

lepton energy fraction ⇒ transferred in the scattering

$0 \leq y \leq 1$

We can now write the cross section as a function of W_1 and W_2

$$L_{\mu\nu} = \frac{1}{2} \times 4\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} kk'\right)$$

spin average

When contracting with $W^{\mu\nu}$ we can exploit the fact that $q^\mu L_{\mu\nu} = 0$

$$\Rightarrow L_{\mu\nu}W^{\mu\nu} = 2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}kk'\right)\left(-g^{\mu\nu}W_1 + \frac{W_2}{M^2}p^\mu p^\nu\right)$$

$$= 2\left(2kk' - 4kk'\right)(-W_1) + \frac{2W_2}{M^2}\left(2pk\,pk' - M^2 kk'\right)$$

$$= 4kk' W_1 + \frac{2W_2}{M^2}\left(2(pk)(pk') - M^2 kk'\right)$$

we now work in the lab. frame

$$\Rightarrow L_{\mu\nu}W^{\mu\nu} = 4EE'(1-\cos\theta)W_1 + \frac{2W_2}{M^2}\left(2M^2EE' - M^2EE'(1-\cos\theta)\right)$$

$$= 4EE'\left(W_2\cos^2\theta/2 + 2W_1\sin^2\theta/2\right)$$

The ensuing cross section is

normalization for $W_{\mu\nu}$

$$d\sigma = \frac{1}{4\sqrt{(kP)^2 - m^2M^2}} \; \frac{e^4}{q^4} \; L_{\mu\nu}W^{\mu\nu}\,(4\pi M) \; \frac{d^3k'}{2E'(2\pi)^3}$$

flux factor

$$\Rightarrow \quad \left.\frac{d\sigma}{dE'd\Omega}\right|_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}} \left\{ W_2(\nu,q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu,q^2)\sin^2\frac{\theta}{2} \right\}$$

Comments : the unknown structure of the proton is embodied in the two functions W_1, W_2
the rest comes from the "upper" part of the diagram that we could compute

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There is an additional integration over E' (not fixed anymore by mass shell condition on X)

Let us go back to the hadronic tensor $W_{\mu\nu}$

$$W_{\mu\nu} = W_1\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{M^2}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right)$$

We can define projectors on longitudinal and transverse polarizations for the photon

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choose P and q' as Sudakov vectors $\quad q' = q + xP \quad q'^2 = 0$

$$\varepsilon = \alpha P + \varepsilon_T + \beta q' \qquad P_T^{\mu\nu} = -g^{\mu\nu} + \frac{p^\mu q'^\nu + p^\nu q'^\mu}{pq'} = -g^{\mu\nu} + \frac{p^\mu q^\nu + p^\nu q^\mu}{pq} - q^2\frac{p^\mu p^\nu}{(pq)^2}$$

transverse projector with respect to P, q'

$$P_L^{\mu\nu} \sim p^\mu p^\nu$$

Applying the above projectors on $W_{\mu\nu}$ we get

$W_{\mu\nu}P_T^{\mu\nu} = \;?$ we can check that the term in W_2 vanishes

$$-g^{\mu\nu}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right) = -\left(p - \frac{pq}{q^2}q\right)^2$$

$$\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left[\frac{p^\mu q^\nu + p^\nu q^\mu}{pq} - q^2\frac{p^\mu p^\nu}{pq^2}\right] = \left[p^\nu - \frac{pq}{q^2}\left(q^\nu + \cancel{\frac{q^2}{pq}p^\nu} - \cancel{q^2\frac{p^\nu}{pq}}\right)\right]$$

$$\Rightarrow \quad W_{\mu\nu}P_T^{\mu\nu} \sim W_1$$

(*)

$$W_{\mu\nu}P_L^{\mu\nu} = \frac{pq^2}{q^2}\left(W_1 + \frac{W_2}{M^2}\left(M^2 - \frac{pq^2}{q^2}\right)\right) \sim W_1 + W_2\left(1 - \frac{\nu^2}{q^2}\right)$$

(*) Other notation

$\nearrow$ dimensionless

$w_1 = F_1/M$

$w_2 = F_2 \frac{M}{pq}$ (better because we neglect M^2 wrt pq)

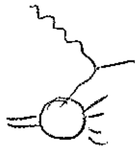
$\Rightarrow \quad W_{\mu\nu} P_L^{\mu\nu} \sim F_2 - 2x F_1$

Parton model

When Q^2 becomes large the virtual photon acts as a probe of the proton structure

The proton behaves as a collection of weakly interacting constituents, called "partons"

The characteristic time of interaction between the partons is much longer than the interaction time of the photon with the single partons



⇒ the picture is that of an incoherent (classical) scattering off the constituent partons

Let us go back to the expression for the differential ep cross section

$$\left.\frac{d\sigma^{ep}}{d\Omega\, dE'}\right|_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}} \left\{ W_2(\nu,q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu,q^2)\sin^2\frac{\theta}{2} \right\}$$

for the scattering cross section off massive charged particles we have derived

$$\left.\frac{d\sigma^{em}}{d\Omega}\right|_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}} \frac{E'}{E} \left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2 \right)$$

$\nu = \frac{pq}{M}$

$q^2 = -4EE'\sin^2\frac{\theta}{2}$

that we can also write as

$$\left.\frac{d\sigma^{em}}{d\Omega\, dE'}\right|_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}} \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2} \right) \delta\left(\nu + \frac{q^2}{2M}\right)$$

↳ $E-E'$]

(the factor E'/E is restored as solution after integration

$\nu + \frac{q^2}{2M} = E - E' - \frac{4EE'\sin^2\frac{\theta}{2}}{2M}$

$= E - E'\left(1 + \frac{2E}{M}\sin^2\frac{\theta}{2}\right)$

⇒ we conclude that for pointlike particles

$$W_2 = \delta\left(\nu + \frac{q^2}{2M}\right)$$

$$2W_1 = -\frac{q^2}{2M^2}\,\delta\left(\nu + \frac{q^2}{2M}\right)$$

At this point it is better to introduce the dimensionless structure functions

$$F_1 = MW_1 = \frac{Q^2}{4M\nu}\,\delta\left(1 - \frac{Q^2}{2M\nu}\right)$$

$$F_2 = \frac{pq}{M}W_2 = \nu W_2 = \delta\left(1 - \frac{Q^2}{2M\nu}\right)$$

that have the nice feature that they depend only on $x = \frac{Q^2}{2M\nu}$ and not on Q^2 and ν separately

Note that this behavior is not the same as in the elastic form factors, that depend on Q^2 and on an additional scale related to the size of the proton. Experimentally it was indeed observed that the structure functions depend (approximately) only on x and not on Q^2

The parton model suggest that the structure functions should be written as

$$F_i(x,Q^2) \sim \sum_a \int_0^1 dz\, f_a(z)\, F_{i,\text{parton}}\left(\frac{x}{z}, Q^2\right)$$

↳ probability of extracting a parton with momentum fraction z

$$x = \frac{Q^2}{2Pq} \qquad p = zP$$

$$x_{\text{parton}} = \frac{Q^2}{2pq} = \frac{1}{z}x$$

⇓ incoherent sum over the scattering of different partons and momentum fractions

Using the previous results we get

$$F_2(x,Q^2) = x \sum_a \overset{\text{charge in unit of the electric charge}}{e_a^2}\, f_a(x)$$

$$F_L = F_2 - 2xF_1 = 0 \; * \qquad \text{CALLAN GROSS RELATION}$$

<u>Spin $\frac{1}{2}$ quarks cannot absorb longitudinally polarized photons</u>

* The scaling invoked for F_1 is nontrivial. Better to derive Callan-Gross relation using full matrix element (exercise 9.3 Halzen-Martin)

In fact

$$\frac{F_1(x)}{F_2(x)} = \frac{W_1(Q^2,\nu)}{W_2(Q^2,\nu)}\frac{M}{\nu} \qquad (m = xM)$$

$$= \frac{Q^2}{4m^2}\frac{M}{\nu} = \frac{Q^2}{2M\nu}\frac{1}{2x^2} = \frac{1}{2x}$$

In the parton model the structure function F_2 is directly related to the parton content of the proton

$$\frac{1}{x} F^{ep}(x) = \left(\frac{2}{3}\right)^2 \left(u^p(x) + \bar{u}^p(x)\right) + \left(\frac{1}{3}\right)^2 \left(d^p(x) + \bar{d}^p(x)\right) + \left(\frac{1}{3}\right)^2 \left(s^p(x) + \bar{s}^p(x)\right)$$

where we have neglected the charm and bottom contribution,

At the same time, we can probe the NEUTRON structure

$$\frac{1}{x} F^{en}(x) = \left(\frac{2}{3}\right)^2 \left(u^n(x) + \bar{u}^n(x)\right) + \left(\frac{1}{3}\right)^2 \left(d^n + \bar{d}^n\right) + \left(\frac{1}{3}\right)^2 \left(s^n(x) + \bar{s}^n(x)\right)$$

SU(2) symmetry implies that

$$u^p(x) = d^n(x) \equiv u(x)$$
$$d^p(x) = u^n(x) \equiv d(x)$$
$$s^p(x) = s^n(x) = s(x)$$

This is a very good approximation. Note however that $\bar{u} \neq \bar{d}$ (since $u > d$ in the proton, also the presence of $\bar{u}$ is affected)

Momentum sum rule

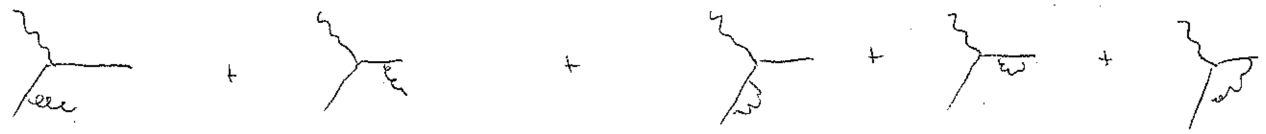
By integrating over x and summing over the flavor of all the partons we should recover the total momentum P of the proton

$$\sum_i \int_0^1 dx\, x f_i(x) = \int_0^1 dx\, x \left(u(x) + \bar{u}(x) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x)\right) = 1$$

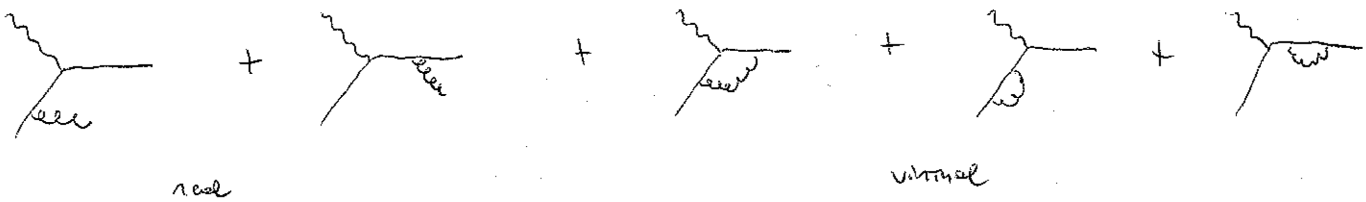
However, experimentally, we find that only about 50% of the proton momentum is accounted for by the quarks $\Rightarrow$ the rest is accounted for by the GLUONS

But gluons do not directly interact with the virtual photon

$\Rightarrow$ we need to consider RADIATIVE CORRECTIONS TO THE PARTON MODEL



We have to consider the following diagrams



Does the naive parton model picture survives after the inclusion of radiative corrections?

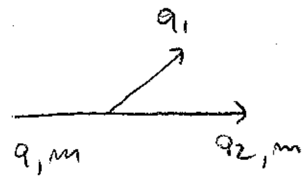
To answer this question we have to talk of INFRARED (IR) singularities

IR singularities

They may originate in theories with massless particles and are of two kinds

SOFT (energy of a massless particle vanish $q_\mu \to 0$)

COLLINEAR $\underline{p_i} \parallel \underline{p_j}$ (leads to singularities when $m=0$)



Define on shell energy transfer as

$$\nu = \omega(\underline{q_1}) + \omega(\underline{q_2}) - \omega(\underline{q_1}+\underline{q_2})$$

It can vanish if:

- $q_1 \to 0 \Rightarrow$ soft singularity

- $\underline{q_1} \parallel \underline{q_2}$

$$\nu = \frac{(\omega_1+\omega_2)^2 - \omega^2(\underline{q_1}+\underline{q_2})}{\omega_1+\omega_2+\omega(\underline{q_1}+\underline{q_2})} = \frac{\cancel{\omega_1^2}+\cancel{\omega_2^2}+2\omega_1\omega_2 - (\cancel{m^2}+\cancel{\underline{q_1}^2}+\cancel{\underline{q_2}^2}+2\underline{q_1}\cdot\underline{q_2})}{\omega_1+\omega_2+\omega(\underline{q_1}+\underline{q_2})}$$

$$\propto 2(\omega_1\omega_2 - \underline{q_1}\cdot\underline{q_2}) = 2\, q_1 q_2$$

QED EXAMPLE

propagator $\frac{1}{(p+q)^2-m^2}$ becomes singular if $\cancel{p^2}+2pq-\cancel{m^2} \to 0$

It can happen if $q \to 0$ but also (in the massless limit) when $\underline{q} \parallel \underline{p}$

$\boxed{\text{KLN THEOREM}}$: In a theory with massless particles IR divergences cancel out in transition rate if we sum over initial and final DEGENERATE states

Makes sense physically : allow the emission of an arbitrarily soft photon or the splitting of two collinear partons

$\Rightarrow$ EXPERIMENTALLY INDISTINGUISHABLE example : TOTAL cross section in $e^+e^- \to$ hadrons

soft and collinear divergences cancel out

Let's now go back to DIS

Here we are inclusive over degenerate FINAL states, but the INITIAL state is fixed

$\Rightarrow$ we have an initial state singularity that remains in the calculation

The singularity shows up in the θ integration $\quad \frac{d\theta^2}{\theta^2} \sim \frac{dk_T^2}{k_T^2}$

We can try to introduce a cut off Q_0

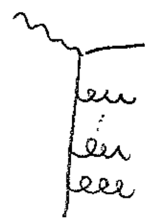
$$\alpha_s \int_{Q_0^2}^{Q^2} \frac{dk_T^2}{k_T^2}$$

but we end up to two problems

$\hookrightarrow \sim 1\,GeV^2$ ($1/Q_0$ could be interpreted as the overall distance of the parton interactions)

1) How to remove the sensitivity to the cut-off?

2) At higher order there are multiple emissions



$\alpha_s^n \log^n \frac{Q^2}{Q_0^2}$

$\alpha_s \log \frac{Q_0^2}{Q_0^2} \sim 1$

$\Rightarrow$ $\boxed{\text{BOTH PROBLEMS CAN BE SOLVED BY THE UNIVERSAL FACTORIZATION OF COLLINEAR SINGULARITIES}}$

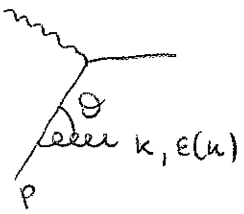
Analogy with the renormalization procedure in QFT

start from $f^{(0)}(z)$ in the naive parton model (bare parton density)

$\Rightarrow$ absorb collinear singularity through its redefinition $\quad f^{(0)}(z) \to f(z)$

The procedure works only if the collinear singularity that we reabsorb is <u>UNIVERSAL</u> (independent of the hard scattering process)

Collinear factorization



phase space $\frac{d^3k}{2w_k} \sim dw_k\, w_k\, d\cos\theta\, d\phi$

↳ $d\theta^2$

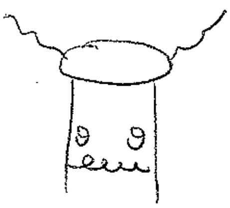
propagator $\frac{1}{(p-k)^2} = \frac{1}{-2pk} \sim \frac{1}{1-\cos\theta} \sim \frac{1}{\theta^2}$

Vertex

If we work in a PHYSICAL GAUGE $\quad p\cdot\varepsilon(k) \sim \theta \quad$ (because ε is a physical (transverse) polarization vector)

⇒ SQUARING THE MATRIX ELEMENT WE GET

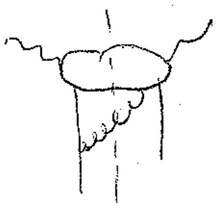
DIRECT



from vertices

$\theta\,\theta \left(\frac{1}{\theta^2}\right)^2$ ← from the (squared) propagator $\rightarrow \frac{d\theta^2}{\theta^2}$ logarithmic singularity

INTERF.

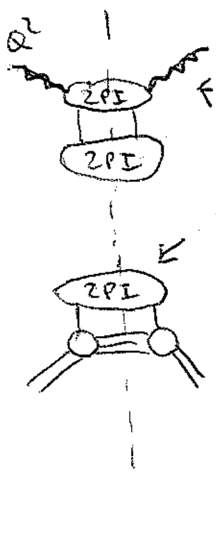


$\theta \cdot \frac{1}{\theta^2}$ not sufficiently singular

(but must be included when working in covariant gauges!)

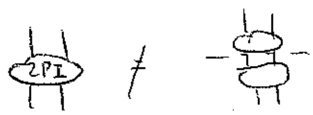
⇒ <u>ONLY DIRECT CONTRIBUTIONS CAN LEAD TO COLLINEAR SINGULARITIES IN A PHYSICAL GAUGE</u>

Decompose cross sections



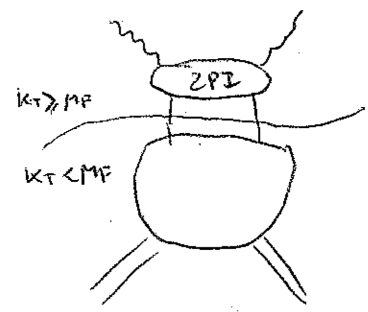
← PROCESS DEPENDENT

LEAD TO COLL. SINGULARITIES BUT <u>UNIVERSAL</u>



cannot be disjoint by cutting only two lines

Introduce factorization scale μ_F



$$\sigma(P,Q^2) = \sum_a \int_0^1 dz\, f_a(z, \mu_F^2)\; \hat{\sigma}_a(zP;\, \alpha_s(Q^2);\, \mu_F)$$

↙ process independent parton density ⇒ now depends on the scale μ_F! NOT COMPUTABLE IN PERTURBATION THEORY BUT UNIVERSAL!

↘ properly subtracted partonic cross section. PROCESS DEPENDENT BUT COMPUTABLE IN PERTURBATION THEORY

μ_F should be chosen of the order of Q otherwise new $\log \frac{Q^2}{\mu_F^2}$ spoil perturbative convergence

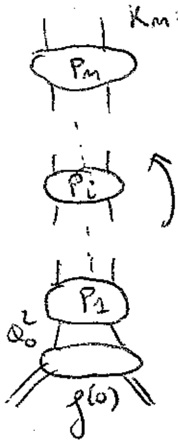
DGLAP EQUATIONS

Parton densities cannot be computed in perturbation theory but their scale dependence can, since it comes from the (universal) properties of collinear singularities

$K_M = x_M P + K_{TM}$ $\qquad P_i = $ 2PI kernel

$K_i = x_i P + K_{Ti}$

$$f(x,Q^2) = \sum_M$$



→ fraction of the transferred longitudinal momentum

DOMINANT CONTRIBUTION COMES FROM STRONGLY ORDERED REGION

$$Q^2 > K_{TM}^2 > \cdots K_{T1}^2 > Q_0^2$$

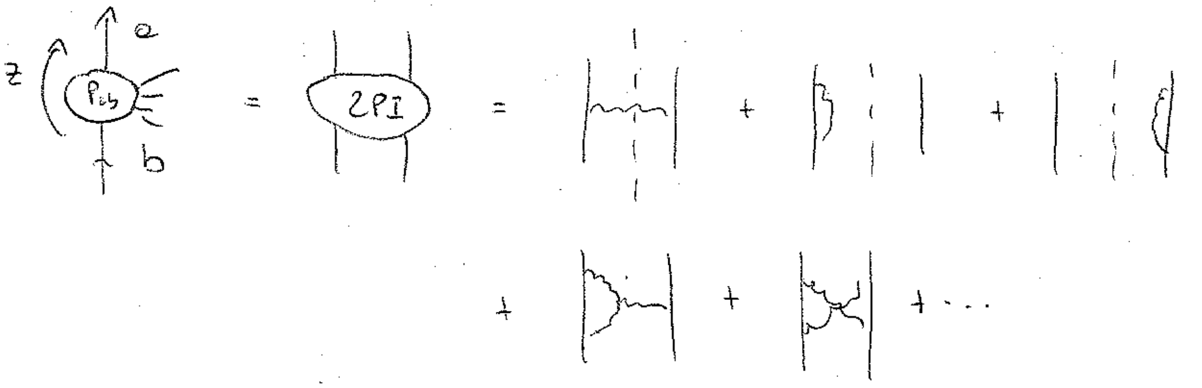
$$\simeq f^{(0)}(x) + \int_{Q_0^2}^{Q^2} \frac{dK_{TM}^2}{K_{TM}^2} \int_x^1 \frac{dz_M}{z_M} P_M(\alpha_s(K_{TM}^2), z_M)\, f\left(x/z_M,\, K_{TM}^2\right)$$

↑ ITERATIVE PICTURE

⇒ TAKING DERIVATIVE WITH RESPECT TO Q^2

$$\boxed{\frac{d f(x,Q^2)}{d\ln Q^2} = \int_x^1 \frac{dz}{z}\, P(\alpha_s(Q^2), z)\, f\left(\frac{x}{z}, Q^2\right)}$$

first order integro differential equation



$$= \frac{\alpha_s}{2\pi} P_{ab}^{(LO)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{NLO}(z) + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ab}^{NNLO}(z)$$

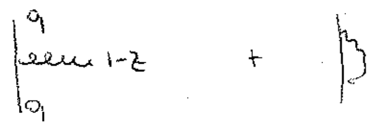
Altarelli Parisi Dokshitzer Gribov-Lipatov

Curci Furmanski Petronzio

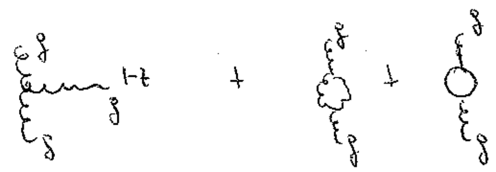
Moch Vermaseren Vogt

DGLAP AT LO

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)$$

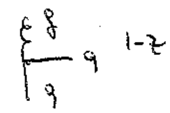


$$P_{gg}(z) = 2C_A\left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right) + \delta(1-z)\,\frac{1}{6}\,(11 C_A - 2 n_F)$$



$$P_{qg}(z) = T_R\left(z^2 + (1-z)^2\right)$$

$$P_{gq}(z) = C_F\,\frac{1+(1-z)^2}{z}$$



Symmetries

$$P_{gg}^{real}(z) = P_{gg}^{real}(1-z) \qquad P_{qq}^{real}(z) = P_{gq}^{real}(1-z)$$

$$P_{qg}(z) = P_{qg}(1-z)$$

Soft behavior $\quad P_{qq} \underset{z\to 1}{\longrightarrow} 2C_F\left(\frac{1}{1-z}\right)_+ \qquad P_{gg} \underset{z\to 1}{\longrightarrow} 2C_A\left(\frac{1}{1-z}\right)_+$

$$\frac{df_q(x,Q^2)}{d\ln Q^2} = P_{qq} \otimes f_q + P_{q\bar q} \otimes f_{\bar q} + P_{qg} \otimes f_g$$

$$\frac{df_g(x,Q^2)}{d\ln Q^2} = P_{gg} \otimes f_g + P_{g\bar q} \otimes f_{\bar q} + P_{gq} \otimes f_q$$

CONVOLUTION $\leftrightarrow$ LONGITUDINAL MOMENTUM CONSERVATION

$$(f \otimes g)(x) = \int_x^1 \frac{dz}{z} f(z) g\left(\frac{x}{z}\right) \qquad f_N = \int_0^1 f(x)\, x^{N-1} \quad \rightarrow \text{MELLIN TRANSFORM}$$

$$\int_0^1 (f\otimes g)(x)\, x^{N-1} = \int_0^1 dx\, x^{N-1} \int_x^1 \frac{dz}{z} f(z) g\left(\frac{x}{z}\right) = \int_0^1 z^{N-1} t^{N-1}\, z\, dt\, \frac{dz}{z} f(z) g(t)$$

$$\frac{x}{z} = t \qquad dx = z\,dt$$

$$= f_N\, g_N$$

$\Rightarrow$ CONVOLUTION PRODUCT BECOMES ORDINARY PRODUCT IN MELLIN SPACE !

$$\frac{df_N^a}{d\ln Q^2} = \gamma_{ab}^N f_N^b \qquad \gamma_{ab}^N(\alpha_s) = \int_0^1 dz\, z^{N-1} P_{ab}(z,\alpha_s)$$

DGLAP eq. becomes an ordinary differential equation

Let us consider for a moment the NON SINGLET channel $q_i - q_j$ (DRIVEN BY P_{qq})

$$\frac{df_N}{d\ln Q^2} = \gamma_{qq}^N(\alpha_s(Q^2))\, f_N(Q^2)$$

$$\frac{df_N}{f_N} = \gamma_{qq}^N(\alpha_s(Q^2))\, d\ln Q^2 \quad \Rightarrow \quad f_N(Q^2) = f_N(Q_0^2)\, e^{\int_{Q_0^2}^{Q^2} \gamma_{qq}^N(\alpha_s(q^2))\, d\ln q^2}$$

$$= f_N(Q_0^2)\, e^{\int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{\gamma_{qq}^N(\alpha)}{\beta(\alpha)} d\alpha}$$

$$\frac{d\alpha}{d\ln q^2} = \beta(\alpha)$$

$$\beta(\alpha) = -\beta_0 \alpha^2 + \dots$$

$$\gamma(\alpha) = \frac{\alpha}{2\pi}\gamma_0 + \dots$$

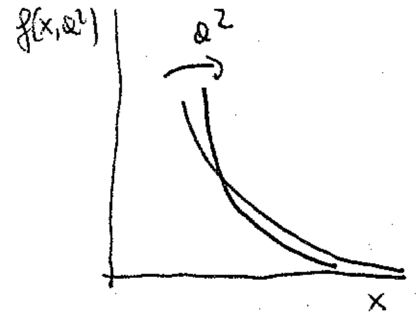
$$e^{\int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{\frac{\gamma_0}{2\pi}\alpha}{-\beta_0\alpha^2} d\alpha} = e^{-\frac{\gamma_0/2\pi}{\beta_0}\int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{d\alpha}{\alpha}} = e^{-\frac{\gamma_0/2\pi}{\beta_0} \log \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}} = \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\frac{\gamma_0}{2\pi\beta_0}}$$

$$\Rightarrow \quad f_N(Q^2) = f_N(Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{\frac{\gamma_0}{2\pi\beta_0}}$$

Scaling violations are

- positive at small x
- slightly negative at large x

$\Rightarrow$ when Q^2 increases the main effect is the shift of partons from larger to smaller x



- <u>large x</u>

At large x the evolution is driven by

$$P_{qq} \sim 2C_a\left(\frac{1}{1-x}\right)_+$$

to compute mellin transform use $\quad x^{N-1} \approx \Theta\left(x-\left(1-\frac{1}{N}\right)\right) = 1-\Theta\left(1-\frac{1}{N}-x\right)$

$$\Rightarrow \quad \int_0^1 dx\,\frac{x^{N-1}-1}{1-x} \sim -\int_0^{1-\frac{1}{N}}\frac{1}{1-x}\,dx = +\log(1-x)\Big|_0^{1-\frac{1}{N}} = -\ln N$$

$$\Rightarrow \quad \gamma_{aa}(N) \sim -2C_a \log N$$

Suppose now that $f(x,Q_0^2) = (1-x)^{a(Q_0^2)}$ at the starting scale Q_0^2

$$\Rightarrow \quad f_N(Q_0^2) \approx \frac{1}{N^{1+a(Q_0^2)}} \quad \text{and} \quad f_N(Q^2) = \left(\frac{1}{N}\right)^{1+a(Q_0^2)}\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-\frac{C_a \ln N}{\pi\beta_0}}$$

$$= \left(\frac{1}{N}\right)^{1+a(Q_0^2)\; + \;\frac{C_a}{\pi\beta_0}\log\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}}$$

$$\Rightarrow \quad f(x,Q^2) \sim (1-x)^{a(Q_0^2)+\frac{C_a}{\pi\beta_0}\log\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}}$$

AT LARGE X THE DISTRIBUTION TENDS TO ZERO FASTER AS Q^2 INCREASES

At small x the DGLAP evolution is driven by the gluon

$$P_{gg} \xrightarrow{x \to 0} \frac{2C_A}{x} \quad \Rightarrow \quad g_N(Q^2) = g_N(Q_0^2) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{\frac{C_A}{\pi \beta_0 (N-1)}}$$

$\Rightarrow$ one can show that this behavior in Mellin space leads to

$$x g(x) \sim \exp \sqrt{\frac{4C_A}{\pi \beta_0} \log \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \log \frac{1}{x}}$$

- STRONG RISE OF THE GLUON DENSITY AT SMALL X
- FASTER THAN ANY POWER OF $\log \frac{1}{x}$ BUT SLOWER THAN ANY POWER OF $\frac{1}{x}$