Advanced Topics in Quantum Information Theory
Solution 1

FS 2012
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Exercise 1.1 The role of initial correlations: beyond CP maps and the Kraus representation

See sheet handed out in the tutorials.

Exercise 1.2 Error Models in the Lindblad Picture

a.) The differential equation to solve is

$$\frac{d}{dt}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} = \gamma_{\sigma_z}\cdot\begin{pmatrix}0 & -2\rho_{12}\\ -2\rho_{21} & 0\end{pmatrix}\ .$$

Writing $\rho \equiv \vec{\rho} = (\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})^T$ this becomes an ordinary differential equation

$$\frac{d}{dt}\vec{\rho} = \gamma_{\sigma_z}\cdot\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\vec{\rho}\ .$$

For $\vec{\rho}(0) = (r_{11}, r_{12}, r_{21}, r_{22})^T$ the solution is

$$\rho(t) = \begin{pmatrix}r_{11} & r_{12}\exp(-2\gamma_{\sigma_z}t)\\ r_{21}\exp(-2\gamma_{\sigma_z}t) & r_{22}\end{pmatrix}\ ,$$

and for $t \to \infty$ it becomes

$$\lim_{t\to\infty}\rho(t) = \begin{pmatrix}r_{11} & 0\\ 0 & r_{22}\end{pmatrix}$$

for any $\gamma_{\sigma_z} > 0$. The generalization to K qubits is straightforward, and obviously there can not be any decoherence free subspace. But since the diagonal elements stay invariant under total dephasing, K qubits can be used to store K classical bits.

b.) The differential equation to solve is

$$\frac{d}{dt}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} = \gamma\cdot\begin{pmatrix}2\rho_{22}-2\rho_{11} & -4\rho_{12}\\ -4\rho_{21} & 2\rho_{11}-2\rho_{22}\end{pmatrix}\ .$$

Writing $\rho \equiv \vec{\rho} = (\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})^T$ this becomes an ordinary differential equation

$$\frac{d}{dt}\vec{\rho} = \gamma\cdot\begin{pmatrix}-2 & 0 & 0 & 2\\ 0 & -4 & 0 & 0\\ 0 & 0 & -4 & 0\\ 2 & 0 & 0 & -2\end{pmatrix}\vec{\rho}\ .$$

For $\vec{\rho}(0) = (r_{11}, r_{12}, r_{21}, r_{22})^T$ the solution is

$$\rho(t) = \begin{pmatrix} \frac{1}{2}\left[(r_{11}+r_{22}) + (r_{11}-r_{22})\exp(-4\gamma t)\right] & r_{12}\exp(-4\gamma t) \\ r_{21}\exp(-4\gamma t) & \frac{1}{2}\left[(r_{11}+r_{22}) + (r_{22}-r_{11})\exp(-4\gamma t)\right] \end{pmatrix}$$

and for $t \to \infty$ it becomes

$$\lim_{t\to\infty} \rho(t) = \begin{pmatrix} \frac{1}{2}(r_{11}+r_{22}) & 0 \\ 0 & \frac{1}{2}(r_{11}+r_{22}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

for any $\gamma > 0$. The generalization to K qubits is straightforward, and obviously there can not be any decoherence free subspace. In the lecture we have seen that the Lie algebra corresponding to total decoherence of K qubits is $\mathcal{L} = \mathrm{su}(2^K)$ and by that, there is no decoherence free subspace.

c.) The differential equation to solve is

$$\frac{d}{dt}\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} = \gamma \cdot A \ ,$$

where

$$\begin{aligned}
A_{11} &= -4\rho_{11} + 2\rho_{22} + 2\rho_{23} + 2\rho_{33} \\
A_{12} &= -6\rho_{12} - 2\rho_{13} + 2\rho_{24} + 2\rho_{34} \\
A_{13} &= -2\rho_{12} - 6\rho_{13} + 2\rho_{24} + 2\rho_{34} \\
A_{14} &- 12\rho_{14} \\
A_{21} &= -6\rho_{21} - 2\rho_{31} + 2\rho_{42} + 2\rho_{43} \\
A_{22} &= 2\rho_{11} - 4\rho_{22} - 2\rho_{23} - 2\rho_{32} + 2\rho_{44} \\
A_{23} &= 2\rho_{11} - 2\rho_{22} - 4\rho_{23} - 2\rho_{33} + \rho_{44} \\
A_{24} &= 2\rho_{12} + 2\rho_{13} - 6\rho_{24} - 2\rho_{34} \\
A_{31} &= -2\rho_{21} - 6\rho_{31} + 2\rho_{42} + 2\rho_{43} \\
A_{32} &= 2\rho_{11} - 2\rho_{22} - 4\rho_{32} - 2\rho_{33} + 2\rho_{44} \\
A_{33} &= 2\rho_{11} - 2\rho_{23} - 2\rho_{32} - 4\rho_{33} + 2\rho_{44} \\
A_{34} &= 2\rho_{12} + 2\rho_{13} - 2\rho_{24} - 6\rho_{34} \\
A_{41} &= -12\rho_{41} \\
A_{42} &= 2\rho_{21} + 2\rho_{31} - 6\rho_{42} - 2\rho_{43} \\
A_{43} &= 2\rho_{21} + 2\rho_{31} - 2\rho_{42} - 6\rho_{43} \\
A_{44} &= 2\rho_{22} + 2\rho_{23} + 2\rho_{32} + 2\rho_{33} - 4\rho_{44}
\end{aligned}$$

Writing

$$\rho \equiv \vec{\rho} = (\rho_{11}, \rho_{12}, \rho_{13}, \rho_{14}, \rho_{21}, \rho_{22}, \rho_{23}, \rho_{24}, \rho_{31}, \rho_{32}, \rho_{33}, \rho_{34}, \rho_{41}, \rho_{42}, \rho_{43}, \rho_{44})^T$$

this becomes an ordinary differential equation

$$\frac{d}{dt}\vec{\rho} = \gamma \cdot B\vec{\rho}$$

with B determined by A. For

$$\vec{\rho}(0) = (r_{11}, r_{12}, r_{13}, r_{14}, r_{21}, r_{22}, r_{23}, r_{24}, r_{31}, r_{32}, r_{33}, r_{34}, r_{41}, r_{42}, r_{43}, r_{44})^T$$

and $t \to \infty$ the solution is (e.g. using Mathematica)

$$\begin{aligned}
\rho_{11}(\infty) &= \frac{1}{6}(2r_{11} + r_{22} + r_{23} + r_{32} + r_{33} + 2r_{44}) \\
\rho_{12}(\infty) &= \rho_{13}(\infty) = \rho_{14}(\infty) = \rho_{21}(\infty) = 0 \\
\rho_{22}(\infty) &= \frac{1}{6}(r_{11} + 2r_{22} - r_{23} - r_{32} + 2r_{33} + r_{44}) \\
\rho_{23}(\infty) &= \frac{1}{6}(r_{11} - r_{22} + 2r_{23} + 2r_{32} - r_{33} + r_{44}) \\
\rho_{24}(\infty) &= \rho_{31}(\infty) = 0 \\
\rho_{32}(\infty) &= \frac{1}{6}(r_{11} - r_{22} + 2r_{23} + 2r_{32} - r_{33} + r_{44}) \\
\rho_{33}(\infty) &= \frac{1}{6}(r_{11} + 2r_{22} - r_{23} - r_{32} + 2r_{33} + r_{44}) \\
\rho_{34}(\infty) &= \rho_{41}(\infty) = \rho_{42}(\infty) = \rho_{43}(\infty) = 0 \\
\rho_{44}(\infty) &= \frac{1}{6}(2r_{11} + r_{22} + r_{23} + r_{32} + r_{33} + 2r_{44})
\end{aligned}$$

Note that this is the solution written in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In the basis $\{|00\rangle, |11\rangle, |01+10\rangle, |01-10\rangle\}$ it becomes

$$\begin{pmatrix} \frac{1-a}{3} & 0 & 0 & 0 \\ 0 & \frac{1-a}{3} & 0 & 0 \\ 0 & 0 & \frac{1-a}{3} & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

where $a = \frac{1}{2}(r_{22} - r_{23} - r_{32} + r_{33})$ is the support of the initial density matrix on the singlet subspace $|01-10\rangle\langle 01-10|$. Thus we see that the singlet subspace stays invariant and the triplet subspace goes to a normalized identity. This is in accordance with the results from the lecture, where we have seen that the singlet subspace is the decoherence free subspace.

Exercise 1.3 Collective Decoherence

The Lindblad operators for collective decoherence of K qubits are given by $\sigma_x \otimes \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_x \otimes \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2 + \ldots + \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2 \otimes \sigma_x$ and the same for y and z. Instead of solving the corresponding Lindblad equation, we can use the representation theoretic results from the lecture, which tell us that the decoherence free subspace is given by $\text{span}\{\pi|01-10\rangle^{\otimes \frac{K}{2}}, \pi \in S_K\}$.