Üb 2.3 : Lie'sche Ableitung

We define the Lie derivative of a tensor field $T \in T^r_s M$ as

$$(L_X T)(X_1, \dots, X_r, \omega^1, \dots, \omega^s)$$

$$:= X\big(T(X_1, \dots, X_r, \omega^1, \dots, \omega^s)\big)$$

$$- \sum_{k=1}^{r} T(\dots, L_X X_k, \dots, \omega^1, \dots, \omega^s)$$

$$- \sum_{l=1}^{s} T(X_1, \dots, X_r, \dots, L_X \omega^l, \dots).$$

In the local coordinate we have:

$$L_X X_k \overset{(X_k \equiv \frac{\partial}{\partial x^k} \equiv \partial_k)}{=} [X, \partial_k] \underset{\text{(we the Einstein's summation conversion)}}{=} [X^j \partial_j, \partial_k]$$

$$= X^j \frac{\partial^2(\cdot)}{\partial x^j \partial x^k} - \frac{\partial X^j}{\partial x^k} \partial_j - X^j \frac{\partial^2(\cdot)}{\partial x^k \partial x^j}$$

$$= - \frac{\partial X^j}{\partial x^k} \partial_j$$

such that

$$T(\dots, L_X X_k, \dots, \omega^1, \dots, \omega^s) = -\sum_{j=1}^{r} \partial_k X^j \, T(\dots, X_j, \dots, \omega^1, \dots, \omega^s)$$

and we have :

$$L_X \omega^l = L_X dx^l \overset{\substack{\text{(see below)} \\ (*)}}{=} d L_X x^l = d\left(X^j \frac{\partial x^l}{\partial x^j}\right)$$

$$= d X^l = \frac{\partial X^l}{\partial x^j} dx^j$$

such that

$$T(X_1, \dots, X_r, \dots, L_X \omega^l, \dots) = \partial_j X^l \, T(X_1, \dots, X_r, \dots, \omega^j, \dots$$

Finally we have :

$$(L_X T)^{i_1, \dots, i_s}_{j_1, \dots, j_r} = X^j \partial_j T^{i_1, \dots, i_s}_{j_1, \dots, j_r}$$

$$+ \sum_{k=1}^{r} (\partial_k X^j) \, T^{i_1, \dots, i_s}_{j_1, \dots, j, \dots, j_r}$$

$$- \sum_{l=1}^{s} (\partial_j X^l) \, T^{i_1, \dots, j, \dots, i_s}_{j_1, \dots, j_r}$$

Note that in $(*)$ we used the fact that $L_X d = d L_X$.
In order to show this we want first to show that

$$L_X \omega = (i_X \circ d)(\omega) + (d \circ i_X)(\omega) \qquad (1)$$

Let us consider first the simple case where $\omega = f \in \Omega^0(M)$ is a function (0-form). We have

$$i_X df + d \, i_X f = i_X df = df(X) = X(f) = L_X f \, !$$

In the case of a 1-form $\omega \in \Omega(M)$, we can write

$\omega = u\,dv$ where u, v are smooth functions.

We first evaluate:

$$i_X\, d(\omega) = i_X\, d(u\,dv) = i_X(du \wedge dv) + i_X(u \underbrace{d^2 v}_{=0})$$

$$= i_X(du \wedge dv) = (i_X\, du)\wedge dv - du \wedge (i_X\, dv)$$

$$= du(X)\wedge dv - du \wedge dv(X)$$

$$= (X(u))\, dv - du\, (X(v)). \qquad (a)$$

and

$$d(i_X\, \omega) = d(i_X\, u\, dv) = d(u\, dv(X)) = d(u\, X(v))$$

$$= (du)\, X(v) + u\, d X(v) \qquad (b)$$

We take now the sum of (a) and (b), and we get

$$i_X\, d(\omega) + d(i_X \omega) = X(u)\, dv + u\, d(X(v)). \quad (c)$$

We evaluate now

$$L_X\, \omega = L_X(u\, dv) = (L_X\, u)\, dv + u\, L_X(dv)$$

$$\overset{(**)}{=} X(u)\, dv + u\, d(L_X\, v) = X(u)\, dv + u\, d X(v)$$

this is nothing but (c)!

In the equality (**), we used the fact that $L_X(dv) = d(L_X\, v)$. (2)
We show this as follows:

We have for the function f:

$$(L_X\, df)(Y) = X(df(Y)) - df(L_X Y)$$

$$= X(Y(f)) - \underbrace{df([X,Y])}_{[X,Y](f)}$$

$$= \cancel{X(Y(f))} - \cancel{X(Y(f))} + Y(X(f))$$

$$= Y(X(f))$$

$$= d(X(f))(Y) = (d\, L_X)(f)$$

Taking $f = v$, $(**)$ is shown.

Let us now consider the general case of a k-form $\omega \in \Omega^k(M)$

We can decompose ω as:

$$\omega = \sum_{i_1,\dots,i_k} \omega_{i_1,\dots,i_k}\, dx^{i_1}\wedge\dots\wedge dx^{i_k} = \sum_I \omega_I\, dx^I \quad (I \equiv (i_1,\dots,i_k))$$

We can also write:

$$\omega = \sum_I \alpha^I \wedge \beta^I$$

where $\alpha^I = \omega_I\, dx^{i_1}$ and $\beta^I = dx^{i_2}\wedge\dots\wedge dx^{i_k}$.

$(\alpha \in \Omega^1(M))$ $\quad$ $(\beta^I \in \Omega^{k-1}(M))$

On one hand we have:

$$L_X(\alpha^I \wedge \beta^I) = L_X \alpha^I \wedge \beta^I + \alpha^I \wedge L_X \beta^I$$

We have shown that the relation (1) holds for a one-form, i.e.

$$L_X \alpha^I = d\, i_X \alpha^I + i_X\, d\alpha^I$$

The $(k-1)$-form β^I can itself be decomposed as a sum of terms $\alpha^{I_2} \wedge \beta^{I_2}$ where α^{I_2} is a 1-form and β^{I_2} is a $(k-2)$-form. We continue this procedure untill we reach $\alpha^{I_{k-1}} \wedge \beta^{I_{k-1}}$ where $\beta^{I_{k-1}}$ is a $(k-(k-1))$-form i.e. $\beta^{I_{k-1}}$ a 1-form! and the relation (1) holds for it

This proves the relation (1) for any k-form ω.

In order to show (2) we simply use (1) and the fact that $d^2=0$. We have:

$$L_X d\omega = i_X \underbrace{d(d\omega)}_{=0} + d\, i_X \omega$$

and

$$d L_X \omega = d(i_X \omega) + \underbrace{d\, d\,(i_X \omega)}_{=0}$$

$$\Rightarrow \quad L_X d\omega = d L_X \omega$$

Q.E.D