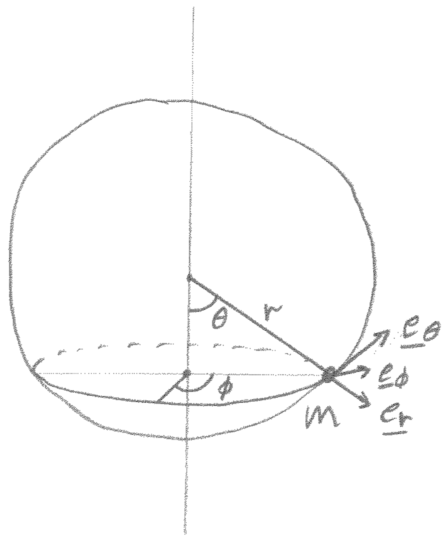
Üb 2.1

The Spherical Pendulum



$\downarrow g$

$$\begin{cases} 0 \leq \theta < \pi \\ 0 \leq \phi < 2\pi \end{cases}$$

$$\begin{cases} x = r \cos\phi \sin\theta \\ y = r \sin\phi \sin\theta \\ z = -r \cos\theta \end{cases}$$

Let us write the Lagrangian of the system:

$$\begin{aligned} L &= T - V(z) \\ &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2} m r^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mgr\cos\theta \\ &\equiv L(\theta, \dot{\theta}, \dot{\phi}) \end{aligned}$$

Notice that L does not depend on ϕ explicitely, then the following quantity is a constant of the motion (as we will see later)

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} \sin^2\theta \quad (1)$$

(angular momentum in the $\underline{e}_\phi$ direction)

We derive the Euler-Lagrange equations:

$$m r^2 \ddot{\theta} = m r^2 \dot{\phi}^2 \sin\theta\cos\theta - mgr\sin\theta \quad (2)$$

$$\ddot{\phi} \sin^2\theta + 2\dot{\theta}\dot{\phi} \sin\theta\cos\theta = 0 \quad (3)$$

From the equation (3) we see that $\dot{p}_\phi = 0$! This is the conservation of angular momentum.

From (1) we can write $\dot{\phi} = \dfrac{p_\phi}{m r^2 \sin^2\theta}$ and substitute it in (2). We get

$$\ddot{\theta} = - \frac{\partial V_{eff}}{\partial \theta}$$

with $V_{eff}(\theta) = - \dfrac{g}{r} \cos\theta + \dfrac{J^2}{2 m^2 r^4} \dfrac{1}{\sin^2\theta}$.

Since $\dfrac{\partial L}{\partial t} = 0$, we know that the energy of the system is conserved:

$$\frac{E}{m r^2} = \frac{1}{2} \dot{\theta}^2 + V_{eff}(\theta) .$$

One alternative to get the Hamiltonian function is to perform a Legendre transformation: $\mathcal{H}(p, q) = \sum_i p_i \dot{q}^i - L(q, \dot{q})$

$$\mathcal{H}(p_\phi, p_\theta, \theta, \phi) = p_\phi \dot{\phi} + p_\theta \dot{\theta} - L(\theta, \phi, \dot{\theta}, \dot{\phi})$$

where $p_\phi = \dfrac{\partial L}{\partial \dot{\phi}}$ and $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}}$.

Then we get

$$\mathcal{H}(p_\theta, p_\phi, \theta) = \frac{p_\theta^2}{2 m r^2} + \frac{p_\phi^2}{2 m r^2 \sin^2\theta} - m g r \cos\theta , \quad (4)$$

$$(\mathcal{H} = T + V)$$

with the Hamilton equations of motion:

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} \longrightarrow \quad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{m r^2} ; \quad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{m r^2 \sin^2\theta} , \quad (5$$

$$\dot{p}_i = - \frac{\partial \mathcal{H}}{\partial q^i} \longrightarrow \quad \dot{p}_\theta = - \frac{\partial \mathcal{H}}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{m r^2 \sin^3\theta} - m g r \sin\theta ; \quad \dot{p}_\phi = - \frac{\partial \mathcal{H}}{\partial \phi} = 0 .$$

We want now to look at the system close to the stable equilibrium state (small θ and $\dot{\phi}=0$).

a) From the Euler-Lagrange equations of motion:

From the equation (2) we see that the condition $\dot{\theta}=\dot{\phi}=0$ (equilibrium state)(*) gives us the solution $\theta(\dot{\theta}=0)\equiv\theta_0=0$, $\phi=\phi_0$ (constant).

We introduce now small perturbations:

$$\theta(t)=\theta_0+\delta\theta(t) \Rightarrow \dot{\theta}=\delta\dot{\theta}\,,\ \ddot{\theta}=\delta\ddot{\theta}.$$

$$(\delta\theta \ll 1)$$

The equation (2) now reads

$$m r^2\,\delta\ddot{\theta}=-mgr\,\sin(\delta\theta)$$

expanding for small $\delta\theta$, we get

$$\sin\delta\theta \simeq \delta\theta+\mathcal{O}((\delta\theta)^2)$$

$$m r^2\,\delta\ddot{\theta}=-mgr\,\delta\theta$$

$$\Leftrightarrow \quad \delta\ddot{\theta}+\frac{g}{r}\,\delta\theta=0$$

$$\Leftrightarrow \quad \delta\ddot{\theta}+\omega^2\,\delta\theta=0$$

This is the equation of motion of a harmonic oscillator with the frequency $\omega=\sqrt{\frac{g}{r}}$.

(*) Note that the solution $(\theta_0,\phi_0)=(\pi,\phi_0)$ is also an equilibrium state but it is not stable.

b) From the Hamilton equations of motion:

From the equilibrium condition ($\dot{\theta} = \dot{\phi} = \dot{p}_\theta = \dot{p}_\phi = 0$), we get from (5):

$$\begin{cases} p_\theta = 0 \\ p_\phi = 0 \end{cases} \Rightarrow \begin{cases} \sin\theta = 0 \Rightarrow \theta_0 = 0 \\ \phi = \phi_0 \text{ (constant)}. \end{cases}$$

Introducing small perturbations ($\theta(t) = \theta_0 + \delta\theta(t)$) in (5), we get:

$$\begin{cases} p_\theta = m r^2 \delta\dot{\theta} \\ p_\phi = 0 \end{cases} \qquad \begin{aligned} \dot{p}_\theta &\simeq - m g r\, \delta\theta \\ \dot{p}_\phi &= 0 \end{aligned}$$

and the Hamiltonian now reads:

$$H + mgr = \frac{m r^2}{2} \delta\dot{\theta}^2 + \frac{m g r}{2} \delta\theta^2 = \tilde{H}$$ (taking the zero of the potential energy at $z = -r$)

$$= \frac{m}{2} (r\,\delta\dot{\theta})^2 + \frac{m}{2}\frac{g}{r} (r\,\delta\theta)^2$$

$$\equiv \frac{m\, v^2}{2} + \frac{m\,\omega^2}{2} x^2$$

This is nothing but the Hamilton function of a harmonic oscillator with the frequency $\omega = \sqrt{\frac{g}{r}}$.