Quantum Information Theory
Tips 6

FS 11
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Exercise 6.1 Distance bounds

So far we have introduced two distance measures for quantum states: fidelity and trace distance. The former has the nice property of being invariant under purifications, while the latter has an useful operational meaning: it bounds the probability of distinguishing two quantum states. We can combine the two to obtain funny results.
To solve this exercise, you should

1. think about the form of the purification of a product state (use the Schmidt decomposition);
2. use Uhlmann's theorem (theorem 4.3.8, page 37 of the script);
3. get bounds for the trace distance in terms of fidelity (lemma 4.3.10 on page 40 of the script).

Why is this result useful? Well, in general it is easier to find a way of decoupling a system A from some reference then to prove that A is maximally entangled with a different system A', and one often needs entanglement in quantum protocols. Using these bounds we only need to show the decoupling part to get the maximum error probability of any experiment we can make on the hopefully-entangled state.
Two concrete examples are channel error estimation and work-extraction protocols. We can talk about channels after introducing the Stinespring dilation, and about work extraction when you have time and patience.

Recap of measurements on bipartite states

Suppose there is a state $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ and you want to perfom a measurement represented by the observable $O_A = \sum_y y P_y$ on subsystem A. Here $\{y\}_y$ are the eigenvalues of the operator and the projectors P_y have the form $P_y = \sum_\alpha |y^\alpha\rangle\langle y^\alpha|$, where $\{|y^\alpha\rangle\}_\alpha$ are the eigenvectors associated with eigenvalue y.
On the total system the measurement is represented by $O = O_A \otimes \mathbb{1}_B$. The probability of obtaining the outcome y is given by

$$\mathrm{Pr}_{O,\rho}(y) = \mathrm{Tr}([P_y \otimes \mathbb{1}_B]\rho_{AB})$$

and after the measurement (with outcome y) the state collapses to

$$\rho_B(y) = \frac{\mathrm{Tr}_A([P_y \otimes \mathbb{1}_B]\rho_{AB})}{\mathrm{Pr}_{O,\rho}(y)},$$

where Tr_A is the partial trace over subsystem A.

Exercise 6.2 Bell-type Experiment

We will see later on the semester that Bell experiments show that quantum mechanics produces phenomena that cannot be predicted using local classical probability theory — local hidden variables. For now we will not try to compare the quantum results with what is achievable classically, but simply observe their strangeness.
The setting goes as follows. There are two parties (usually called Alice and Bob) that prepare an entangled two-qubit state. Alice keeps one of the qubits and Bob the other. No matter how far apart they are, their qubits are still entangled.
Now Alice will measure her qubit in a given basis. This will cause Bob's qubit to collapse to some state so that when he measures his qubit the measurement statistics are different from if he had decided to measure his qubit before Alice measured hers.
A well known example is when the state prepared is

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

for a basis $\{|0\rangle, |1\rangle\}$ in A and B.
We will see the simplest case, when Alice performs a measurement in that same basis.
The probability that she gets 0 is

$$\begin{aligned} P_A(0) &= \mathrm{Tr}([|0\rangle\langle 0| \otimes \mathbb{1}_B] \frac{1}{2} (|0\rangle|0\rangle + |1\rangle|1\rangle) (\langle 0|\langle 0| + \langle 1|\langle 1|)) \\ &= \frac{1}{2} \end{aligned}$$

and if she obtained 0 the whole state collapses to

$$\begin{aligned} \rho_{B|A=0} &= 2\mathrm{Tr}_A([|0\rangle\langle 0| \otimes \mathbb{1}_B] \frac{1}{2} (|0\rangle|0\rangle + |1\rangle|1\rangle) (\langle 0|\langle 0| + \langle 1|\langle 1|)) \\ &= |0\rangle\langle 0|, \end{aligned}$$

so that if Bob now measures his qubit in that basis he will always obtain $|0\rangle$. Suppose now that Alice did not tell Bob what she obtained in the measurement. In this case Alice knows that Bob has state $\rho_{B|A=0}$ but from the point of view of Bob his state is

$$\begin{aligned} \rho_B &= \mathrm{Tr}_A(|\psi^+\rangle\langle\psi^+|) \\ &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|). \end{aligned}$$

If Alice and Bob were to bet on the outcome of a measurement on Bob's qubit (in basis $\{|0\rangle, |1\rangle\}$), Alice would bet on 0 and win with 100% probability, while Bob will only win with probability $\frac{1}{2}$, as from his point of view both outcomes are equally likely. In this case the probability distributions on the outcomes of Bob's measurement are

$$P_{B|A=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad P_B = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

It may seem at first sight that something is wrong – how can the same physical qubit be represented by two different density operators and have two different probability distributions on the same physical measurement? The answer is that that qubit is correlated to another system (Alice's qubit) and what we are looking at are the states (and probability distributions) conditioned / not conditioned on an event on that system (Alice's measurement) that is itself random: Alice is equally likely to obtain 0, in which case Bob's state collapses to $|0\rangle\langle 0|$, or 1, when get $|1\rangle\langle 1|$ on Bob's side.
There is one degree of freedom for Bob: he can choose the basis in which to measure his state. In basis $\{|0\rangle, |1\rangle\}$ he will always get 0, but if he chooses basis $\{|+\rangle, |-\rangle\}$, for instance, he may get either result with equal probability. In general, as you will prove, the *more distant* Alice's and Bob's bases are the bigger the uncertainty on Bob's measurement after Alice performed hers.
In this exercise Alice measures her state in an arbitrary basis $\left\{|\alpha\rangle, |\alpha\rangle^\perp\right\}$. We are dealing with a two-state system and a new basis may be defined as a rotation of a known one by an angle α. In our case, $|\alpha\rangle := \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle$ and $|\alpha\rangle^\perp$ corresponds to $|\alpha + \frac{\pi}{2}\rangle$.
To obtain the reduced state on B after the measurement on A when the outcome is known, you just have to apply the rules given above. Then you have to calculate what Bob will obtain when measuring his qubit in the "original" basis $\{|0\rangle, |1\rangle\}$. I suppose you know how to do it in the case of unknown outcome on Alice's side.

Exercise 6.3 Depolarizing channel

In this exercise we will see how to use quantum operations to define channels, which you surely remember from the second exercise series. The essential tools here are trace preserving completely positive maps (TCPMs). You can read all about them on pages 40 to 45 of the script. As the name suggests, TCPMs map positive operators to positive operators and preserve their trace — in particular they map density operators to density operators. The evolution of a system can always be represented by a TCPM: $\rho_{t_1} = \mathcal{E}(\rho_{t_0})$.

Let us examine the TCPM we are given in this exercise,

$$\mathcal{E}_p : \rho \mapsto \frac{p}{2}\mathbb{1} + (1-p)\rho,$$

where ρ is the density operator of a qubit. At first sight we notice that is redistributes the *weight* of the density operator: $1-p$ of it stays as before but p becomes fully mixed.
For now that channel seems maybe a bit abstract and that is why in part *a*) of the exercise we are asked to find a way of *implementing* it, ie. to express is in terms of operators qe know how to deal with — and in the case of qubits these will be Pauli matrices.
We have fo find an operator-sum representation of $\mathcal{E}_p$, meaning that we need to find operators $\{E_k\}_k$ such that

$$\frac{p}{2}\mathbb{1} + (1-p)\rho = \sum_k E_k \rho {E_k}^*.$$

My suggestion is that you start by using the Bloch sphere representation of qubits, $\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$ to express the identity in terms of Pauli matrices and ρ. Be careful with the properties of Pauli operators such as

$$\begin{aligned} {\sigma_i}^2 &= \mathbb{1} \\ [\sigma_i, \sigma_j] :&= \sigma_i\sigma_j - \sigma_j\sigma_i = 2\mathbb{1}\varepsilon_{ijk}\sigma_k, \\ \{\sigma_i, \sigma_j\} :&= \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}. \end{aligned}$$

You should obtain $\mathbb{1} = \frac{1}{2}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)$.
Now insert that in the definition of $\mathcal{E}_p$ and in the end you should get

$$E_{\mathbb{1}} = \sqrt{1 - \frac{3p}{4}}\mathbb{1}, \qquad E_i = \frac{\sqrt{p}}{2}\sigma_i, \quad i = x, y, z.$$

Supposing that (eg. with a quantum computer) we know how to apply the Pauli matrices to a qubit, we are now able to implement $\mathcal{E}_p$.
In part *b*) they ask us what happens to the radius $|\vec{r}|$ of the Bloch vector of a state when we apply $\mathcal{E}_p$. Remember that the pure states lied on the surface of the sphere, with $|\vec{r}| = 1$, while the fully mixed state was in its centre, $|\vec{r}| = 0$. Check what happens to $|\vec{r}|$ (you don't need to use the operator-sum representation) and see what it means in terms of the purity of the state.
We can use TCPMs to describe channels — if you recall, sets of conditional probabilities that define maps from one probability distribution to another. In this case we will define it as

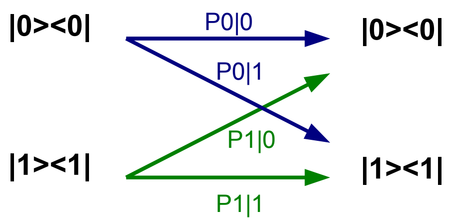

Figure 1: Our channel. Here $Pb|a$ stands for $P_{B|A=a}(b)$.

See what happens when you apply $\mathcal{E}_p(|0\rangle\langle 0|)$ and $\mathcal{E}_p(|1\rangle\langle 1|)$. You will get the conditional probabilities that define the channel from there if you look at the final states as ways of encoding probability distributions on $0, 1$. For instance, the state you obtain from $|0\rangle\langle 0|$ will be of the form of a *classical state* (pages 34-35 of the script),

$$\mathcal{E}_p(|0\rangle\langle 0|) = P_{B|A=0}(0)|0\rangle\langle 0| + P_{B|A=0}(1)|1\rangle\langle 1|.$$

More generally, you can apply $\mathcal{E}_p$ to the initial classical state $q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$ and take the conditional probabilities from there.
Now that you have the channel defined in a classical way you can calculate its capacity just like in the second exercise series. What classical channel does it resemble?