Measurements

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Quantum Information Theory FS11

I'm confused with the notion of measurement. Sometimes, we speak about doing a measurement "represented by the observable $O = \sum \dots$", sometimes we say "do a measurement with respect to the basis $\{\dots\}$" and finally there is also the version "do a measurement with respect to the POVM $\{\dots\}$". I don't see the link between these three methods of measuring a state. Moreover, I have a problem for each of these variants:

1 Observable

In the QM course, we interpret an observable as something to measure, not a measurement itself. For example the momentum p is an observable, and if I measure this observable while the system is in a particular state, I will become a certain value of momentum (the physical quantity). We do not measure the state "with respect to the momentum"... or is it the same?

Yes, it can be a bit fuzzy with all the different conventions. In QIT we call "observable" to an operator that represents a measurement. These operators are of the form $O = \sum_x x P_x$, where

1. $\{x\}_x$ are the possible outcomes of the measurement (for instance if you are measuring a distance then you could have $\{x\} = \{10m, 23km, 3cm\}$) ;
2. $\{P_x\}_x$ are the projectors (operators) that determine the statistics of the measurement: the probability of having outcome x when performing a measurement on state ρ is given by $\mathrm{Tr}(P_x \rho)$, and the state "collapses" to $\rho_x = (P_x \rho P_x)/\mathrm{Tr}(P_x \rho)$;
3. sometimes the name of the observable gives us a hint of what it measures (like calling it P if it measures momentum), but many times we just stick to O;
4. note that because $\{x\}_x$ are the outcomes (what you read in your machine when you measure a state), O is in general not normalised. In fact, the x_x don't even need to be numbers, but more like labels to the different outcomes (see ahead with the giraffes);
5. $\sum_x P_x = \mathbb{1}$.

This is why we say we "measure an observable" or "perform a measurement represented by an observable"" (maybe even "with respect to the observable", though that sounds more like a writing accident), because an observable has everything that is needed to identify and specify a measurement.

2 Measurement w.r.t. a basis

> *For example, in series 12, Alice performs a measurement of the state AS wrt the Bell-basis. My question: how does this work? If she projects the state on one of the Bell- vectors, the state will "collapse"; if the state was in the Bell-vector, Alice gets the measuring value 1, if the state was not in this Bell-vector, she gets 0 but can not try again with the remaining three Bell-vectors, since the state is collapsed...*

Let me give you a simpler example. Instead of the Bell basis (of two qubits), we consider just one qubit (in state ρ) that Alice wants to measure in the computational basis $\{|0\rangle, |1\rangle\}$. It's not like she chooses to project ρ in one of the states (say $|0\rangle$), as you were wondering. What happens when you perform a measurement in that basis is that "the universe" will project the state in one of the states with probabilities given by:

$$\begin{aligned} P_0 &= \mathrm{Tr}(|0\rangle\langle 0|\rho), \\ P_1 &= \mathrm{Tr}(|1\rangle\langle 1|\rho). \end{aligned}$$

If $\rho = a|0\rangle\langle 0| + b|1\rangle\langle 1| + c|0\rangle\langle 1| + d|1\rangle\langle 0|$, then

$$\begin{aligned} P_0 &= \mathrm{Tr}\Big(a|0\rangle\langle 0|0\rangle\langle 0| + b|0\rangle\langle 0|1\rangle\langle 1| + c|0\rangle\langle 0|0\rangle\langle 1| + d|0\rangle\langle 0|1\rangle\langle 0|\Big) \\ &= a \\ P_1 &= \mathrm{Tr}\Big(a|1\rangle\langle 1|0\rangle\langle 0| + b|1\rangle\langle 1|1\rangle\langle 1| + c|1\rangle\langle 1|0\rangle\langle 1| + d|1\rangle\langle 1|1\rangle\langle 0|\Big) \\ &= b \end{aligned}$$

For instance, if ρ is the pure state $|1\rangle\langle 1|$, then $P_0 = 0$ and $P_1 = 1$, which means that if she performs a measurement in the computational basis she will always obtain $|1\rangle$.

Of course you may object to "the universe projects the state at random according to that probability distribution": why does it work like that? I suppose the answer "that's one of the postulates of quantum mechanics - it's just the way the world works" is unsatisfactory. Well, it should be, it's a very strange phenomenon that has been puzzling scientists over the last century. Foundations of quantum mechanics is a branch of quantum information that tries to derive the postulates of quantum theory from more fundamental and "logical" principles. The current state of the art is still a bit green: many times the "basic principles" are often completely abstract and only their inventors find them reasonable and intuitive. Maybe in a couple of years we can teach why it works like this, if you look from the "right" perspective See the lectures by Rob Spekkens if you want to know more (talk of the week 2), and contact Prof. Renner if you would like to work on that—it is a very appealing field, and at least for now less esoteric than string theory.

To make a link to your last question, if we were to represent this measurement as an observable, it could be $O = x_0|0\rangle\langle 0| + x_1|1\rangle\langle 1|$, where x_0 is what Alice sees in her machine when the state is projected to $|0\rangle$ and x_1 what she sees when it goes to $|1\rangle$. Here it is clear why the $\{x\}_x$ are not "numbers": if we used 0 and 1, then we would have $O = |1\rangle\langle 1|$, which is not very representative of our measurement!

3 Measurement w.r.t. a POVM

What is going on here? I have no intuition for this one...

POVMs are just another way of representing a measurement. We forget about what Alice sees in her machine (she can have the machine just saying "*the state collapsed to* $|0\rangle$" instead of "*I got* x_0"; she could have the machine saying "*elephant*" for 0 and "*giraffe*" for 1, for that matter) and focus only on the projectors that determine the statistics of the measurement—the important part.

In our previous example, the POVM that represents Alice's measurement is just the set $M = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

Now suppose that Alice's machine does not work very well: with a small probability p it won't measure the state at all. Then, when she measures the state ρ, three things can happen:

1. with probability p the measurement fails (the machine says "*I failed*") and the final state is
$$\rho_{\text{fail}} = \rho = \mathbb{1}\rho\mathbb{1};$$
2. with probability $1-p$ the measurement works, and then
   - with probability $(1-p)\mathrm{Tr}(|0\rangle\langle 0|\rho)$ she measures "x_0" and the state goes to
   $$\rho_0 = \frac{|0\rangle\langle 0|\rho|0\rangle\langle 0|}{(1-p)\mathrm{Tr}(|0\rangle\langle 0|\rho)} = |0\rangle\langle 0|;$$
   - with probability $(1-p)\mathrm{Tr}(|1\rangle\langle 1|\rho)$ she measures "x_1" and the state goes to
   $$\rho\cdot 1 = \frac{|1\rangle\langle 1|\rho|1\rangle\langle 1|}{(1-p)\mathrm{Tr}(|1\rangle\langle 1|\rho)} = |1\rangle\langle 1|.$$

All in all there are three possible outcomes, which may be represented by an observable

$$\begin{aligned} O' &= p \text{ “fail”} + (1-p) * O \\ &= \text{“fail”}\ p\ \mathbb{1} \\ &\quad + x_0\ (1-p)\ |0\rangle\langle 0| \\ &\quad + x_1\ (1-p)\ |1\rangle\langle 1|, \end{aligned}$$

or by a POVM
$$M' = \{p\ \mathbb{1}, \quad (1-p)\ |0\rangle\langle 0|, \quad (1-p)\ |1\rangle\langle 1|\}\,.$$

The difference is just that in the POVM we get rid of the labels. POVMs have some nice properties and can be used for more things than observables, which come with "labels" that are not exactly a mathematical object. Note that the elements of this (and any) POVM sum up to the identity.