Quantum Information Theory
Series 7

Exercise 7.1 Quantum operations can only decrease distance

a) Given a trace-preserving quantum operation $\mathcal{E}$ and two states ρ and σ, show that

$$\delta\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right) \leq \delta(\sigma, \rho). \tag{1}$$

b) What does Eq. 1 imply about the task of distinguishing quantum states?

Exercise 7.2 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.

a) Show that the *transpose* is a positive operation, and that it is basis-dependent.

b) Let $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a separable state, and let Λ_A be a positive operator on $\mathcal{H}_A$. Show that $\Lambda_A \otimes \mathbb{1}_B$ maps ρ to a positive operator.

The task of characterizing the sets of separable states then reduces to finding a suitable positive map that distinguishes between separable and entangled states.

c) Show that the *transpose* is a probable candidate by testing it on a Werner state (impure singlet)

$$W = x|\psi^-\rangle\langle\psi^-| + (1-x)\,\mathbb{1}/4,$$

where $x \in [0,1]$ and $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. What happens to the eigenvalues of W when you apply the partial transpose on system A: $W^{T_A} := (transpose_A \otimes \mathbb{1}_B)(W)$?

Remark: Indeed, it can be shown that the PPT (*positive partial transpose*) criterion is necessary and sufficient for systems of dimension 2×2 and 2×3.

d) If you are given the description of a quantum state ρ, explain how the partial transpose could be used to determine if a state is entangled.

e) Show that although the partial transpose is basis-dependent, the corresponding eigenvalues are independent under local basis-transformations.

Exercise 7.3 The Choi-Jamiołkowski Isomorphism

The Choi-Jamiołkowski Isomorphism is a useful tool that allows an alternative representation of quantum maps (instead of using the operator sum representation). To do this, we first introduce an alternative representation of operators. Normally, operators that map from the hilbert space $\mathcal{H}_1$ to $\mathcal{H}_2$ are represented as matrices, such as $C = \sum_{ij} c_{ij} |i\rangle_2 \langle j|_1$. We can instead represent them as vectors:

$$|C\rangle\rangle = \sum_{ij} c_{ij} |i\rangle_2 |j\rangle_1. \tag{2}$$

a) Show that $A \otimes B|C\rangle\rangle = |ACB^T\rangle\rangle$, where A, B, and C are operators, and A and B act on the subspaces $\{|i\rangle_2\}_i$ and $\{|j\rangle_1\}_j$ from Eq. 2 respectively. Note that the transpose on B is defined in the basis chosen to represent the operators in Eq. 2.

b) Show that $\mathrm{Tr}_1(|A\rangle\rangle\langle\langle B|) = AB^*$, where A and B are linear maps from the space $\mathcal{H}_1$ to $\mathcal{H}_2$.

c) An operator sum representation of a map acting on the Hilbert space $\mathcal{H}_A$, with output in the space $\mathcal{H}_{A'}$, can be written in the operator-sum notation as:

$$\mathcal{E}(\rho_A) = \sum_k E_k \rho_A E_k^*.$$

Use (a) and (b) to show that there exists a CJ matrix τ such that

$$\mathcal{E}(\rho_A) = \mathrm{Tr}_A(\tau_{A'A}(\mathbb{1}_{A'} \otimes \rho_A^T)).$$

d) Show that τ from (c) can also be written as

$$\tau = (\mathcal{E}_A \otimes \mathbb{1}_{A'})(|\psi^+\rangle_{AA'}\langle\psi^+|),$$

where $|\psi^+\rangle = 1/\sqrt{d}\sum_{i=1}^d |i\rangle_A |i\rangle_{A'} = |\mathbb{1}\rangle\rangle$.

e) What are the CP and TP conditions on τ in the CJ picture?