Quantum Information Theory
Series 5

FS 11
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Exercise 5.1 Measurements on bipartite systems

Consider a state ρ_{AB} in a composed system $\mathcal{H}_A \otimes \mathcal{H}_B$. Suppose you want to perform a measurement described by an observable O_A on subsystem $\mathcal{H}_A$. The operator O_A has eigenvalues (possible outcomes) $\{x\}_x$ and may be written by spectral decomposition as $O_A = \sum_x x P_x$ where $\{P_x\}_x$ are projectors—operators that only have eigenvalues 0 and 1. Show that the measurement statistics (probabilities of obtaining the different outcomes) are the same whether you apply $O_A \otimes \mathbb{1}_B$ on the joint state ρ_{AB} or first trace out the system $\mathcal{H}_B$ and then apply O_A on the reduced state ρ_A.

Exercise 5.2 Distinguishing two quantum states

Suppose you know the density operators of two quantum states $\rho, \sigma \in \mathcal{H}_A$. Then you are given one of the states at random—it may either be ρ or σ with equal probability. The challenge is to perform a single measurement on your state and then guess which state that is.

a) What is your best strategy? In which basis do you think you should perform the measurement? Can you express that measurement using a projector P?

b) Show that the probability of guessing correctly is given by $\frac{1}{2}(1 + \mathrm{Tr}\,[P(\rho - \sigma)])$. Just like in the classical case, that is equivalent to $\frac{1}{2}[1 + \delta(\rho, \sigma)]$, where $\delta(\rho, \sigma)$ is the trace distance between the two quantum states.

Exercise 5.3 Fidelity and Uhlmann's Theorem

Given two states ρ and σ on $\mathcal{H}_A$ with fixed basis $\{|A\rangle_i\}_i$ and a reference Hilbert space $\mathcal{H}_B$ with fixed basis $\{|B\rangle_i\}_i$, which is a copy of $\mathcal{H}_A$, Uhlmann's theorem claims that the fidelity can be written as

$$F(\rho, \sigma) = \max_{|\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle| \,, \tag{1}$$

where the maximum is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ on $\mathcal{H}_A \otimes \mathcal{H}_B$. Let us introduce a state $|\psi\rangle$ as:

$$|\psi\rangle = (\sqrt{\rho} \otimes U_B)\,|\gamma\rangle, \qquad |\gamma\rangle = \sum_i |i\rangle_A \otimes |i\rangle_B, \tag{2}$$

where U_B is any unitary on $\mathcal{H}_B$.

a) Show that $|\psi\rangle$ is a purification of ρ.

b) Argue why every purification of ρ can be written in this form.

c) Use the construction presented in the proof of Uhlmann's theorem to calculate the fidelity between $\sigma' = \mathbb{1}_2/2$ and $\rho' = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ in the 2-dimensional Hilbert space with computational basis.

d) Give an expression for the fidelity between any pure state and the completely mixed state $\mathbb{1}_n/n$ in the n-dimensional Hilbert space.