### Exercise 4.1 Bloch sphere

We keep going over some basics of quantum mechanics. In this exercise we will see how we may represent qubit states as points in a three-dimensional ball.
A qubit is a two level system, whose Hilbert space is equivalent to $\mathbb{C}^2$. The Pauli matrices together with the identity form a basis for $2 \times 2$ Hermitian matrices,

$$\mathcal{B} = \left\{ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{1}$$

where the matrices were represented in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Pauli matrices respect the commutation relations

$$[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i = 2 1 \varepsilon_{ijk}\sigma_k, \tag{2}$$

$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}. \tag{3}$$

We will see that density operators can always be expressed as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{4}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ is the so-called Bloch vector, that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere.

*a*) Using Eq. 4 :

1) Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases, $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|\circlearrowleft\rangle, |\circlearrowright\rangle\}$.
2) Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

*b*) Show that the operator $\rho$ defined in Eq. 4 is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:

1) Hermiticity: $\rho = \rho^\dagger$.
2) Positivity: $\rho \geq 0$.
3) Normalisation: $\mathrm{Tr}(\rho) = 1$.

*c*) Now do the converse: show that any two-level density operator may be written as Eq. 4.

*d*) Check that the surface of the ball is formed by all the pure states.

### Exercise 4.2 Partial trace

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and it is the natural generalisation of the concept of marginal distributions in classical probability theory. Given a density matrix $\rho_{AB}$ on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_A = \mathrm{Tr}_B \rho_{AB}$,

*a*) Show that $\rho_A$ is a valid density operator by proving it is:

1) Hermitian: $\rho_A = \rho_A^\dagger$.
2) Positive: $\rho_A \geq 0$.
3) Normalised: $\mathrm{Tr}(\rho_A) = 1$.

*b*) Calculate the reduced density matrix of system $A$ in the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right), \quad \text{where} \quad |ab\rangle = |a\rangle_{\mathrm{A}} \otimes |b\rangle_{\mathrm{B}}. \tag{5}$$

*c)* Consider a classical probability distribution $P_{XY}$ with marginals $P_X$ and $P_Y$.

1) Calculate the marginal distribution $P_X$ for

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else,} \end{cases} \tag{6}$$

with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.

2) How can we represent $P_{XY}$ in form of a quantum state?
3) Calculate the partial trace of $P_{XY}$ in its quantum representation.

*d)* Can you think of an experiment to distinguish the bipartite states of parts *b)* and *c)*?

### Exercise 4.3 Purification

A decomposition of a state $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is a (non-unique) convex combination of pure states $\rho_A^x = |a_x\rangle\langle a_x|$ such that $\rho_A = \sum_x \lambda_x \rho_A^x$.

*a)* Show that $|\Psi\rangle = \sum_x \sqrt{\lambda_x}\, |a_x\rangle_A \otimes |b_x\rangle_B$ is a purification of $\rho_A$ for *any* orthonormal basis $\{\, |b_x\rangle_B \,\}_x$ of $\mathcal{H}_\mathrm{B}$.

*b)* Show that any two purifications are related by a *local* unitary transformation on the purifying system.

*c)* Mixed states can be formed in many different ways: $\frac{\mathbb{1}_2}{2}$, for instance, may be a mixture of pure states $|+\rangle$ and $|-\rangle$, of $|\downarrow\rangle$ and $|\uparrow\rangle$, etc. Here we will show that the particular mixture that originated a mixed state has no operational meaning: we can generate any decomposition $\{\rho_A^x\}_x$ of a mixed state $\rho_A$ by purifying $\rho_A$ and performing clever measurements on the purifying system.

For $\rho_A$ as defined above, and any purification $|\Phi\rangle$ of $\rho_A$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, find an orthogonal measurement $\{\, M_B^x \,\}_x$ on $\mathcal{H}_B$, such that

$$\lambda_x = \mathrm{Tr}\left[|\Phi\rangle\langle\Phi|(\mathbb{1}_A \otimes M_B^x)\right] \quad \text{and} \quad \rho_A^x = \frac{\mathrm{Tr}_B\left[|\Phi\rangle\langle\Phi|(\mathbb{1}_A \otimes M_B^x)\right]}{\lambda_x}. \tag{7}$$

In this picture $\lambda_x$ is the probability of measuring $x$ and $\rho_A^x$ is the state after such a measurement.