[Coleman 6.4.1; 6.4.2]

• Soliton Quantization •

We give now a simple example of soliton quantization, which will also allow us to introduce the concept of "collective coordinates", which plays an important role in the physics of solitons.

Consider therefore the case of the scalar in 1+1 dimensions, which we discussed in the ~~last~~ lecture. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi)$$

and, provided U has two degenerate minima, we have already found a static solitonic solution $f(x)$. We want to quantize the fluctuations around this classical solution and therefore we write:

$$\phi(x,t) = f(x) + \delta(x,t)$$

where δ has to be considered as a perturbation, we will expand around $\delta = 0$. Notice, from the very beginning, that what follows will be only justified if the corrections from higher order terms are parametrically small, i.e. if these are suppressed by some expansion parameter. The small-δ expansion can be viewed as the semiclassical expansion because it corresponds to consider only those field configurations that are close to the classical trajectory, which in this case is of course nothing but the classical solution $f(x)$. In spite of being denoted sometimes as a "non-perturbative" method, soliton (and instanton) quantization requires a perturbative expansion parameter, just like the ordinary quantization. The only difference among the two cases is that one quantizes around a trivial or a non-trivial classical trajectory.

To proceed, let us expand

$$\delta = \sum_n \xi_n(t)\, \psi_n(x)$$

where the basis of functions $\{\psi_n(x)\}$ is given by the solution of the equation:

$$-\partial_x^2 \varphi_n + U''[\bar\rho]\varphi_n = \omega_n^2 \varphi_n$$

we already encountered this equation, when discussing the classical stability, and we showed the ω_n^2 "energy levels" to be positive. For the time being, the choice of the $\{\varphi_n\}$ as basis is just a choice, it will be justified in a moment. Notice that the φ_n can be chosen to be orthogonal:

$$\int dx\, \varphi_n \varphi_m = \delta_{nm}.$$

We will treat the $\{\varphi_n\}$ as if they formed a discrete set. Actually, the eigenvalues $\{\omega_n\}$ are typically composed of a "zero-mode" $\omega_0 = 0$ followed by a set $\{\omega_n\}$, possibly, and by a continuum. It would not be difficult to generalize the notation to the continuous case, we however mantain our discrete notation for simplicity.

We want first of all to substitute the expansion in the Lagrangian, this will give us the classical lagrangian of the ξ_n variables, that we will afterwards quantize. Notice that the ξ_n's are just the coordinates of our classical system, they will therefore be quantized by the standard rules of quantum mechanics. Let us substitute:

$$L = \int dx\, \mathcal{L} = \int dx \Big[\frac{1}{2}(\partial_0 \delta)^2 - \frac{1}{2}(\partial_x \delta)^2 +$$

$$- \partial_x \delta \partial_x f - \frac{1}{2} \partial_x f \partial_x f - U(f) - \delta U'(f)$$

$$- \frac{1}{2} \delta^2 U''(f) + \mathcal{O}(\delta^3) \Big]$$

where we expanded in powers of δ, as anticipated. We now use integration by parts, and the equations of motion for f, that reads

$$\partial_x^2 f = U'(f),$$

this gives

$$L = -E_S + \int dx \Big[\frac{1}{2}(\partial_0 \delta)^2 - \frac{1}{2} \delta \big[-\partial_x^2 \delta + U''(f) \delta \big] \Big]$$

where

$$E_S = \int dx \, \frac{1}{2}\left[(\partial_x \rho)^2 + U(\rho)\right]$$

is the classical energy of the soliton.

We can now appreciate the reason for the choice of ψ_n, using the equation of ψ_n and the orthogonality, we rewrite L as:

$$L = -E_S + \sum_n \left[\frac{1}{2}\dot{\xi}_n^2 - \frac{1}{2}\omega_n^2 \xi_n^2\right]$$

from which the Hamiltonian

$$H = E_S + \frac{1}{2} P_0^2 + \sum_n \left[\frac{1}{2} P_n^2 + \frac{1}{2}\omega_n^2 \xi_n^2\right]$$

where we isolated the "zero mode" coordinate ξ_0, with $\omega_0 = 0$, from the other modes with $\omega_n^2 > 0$. Notice that the existence of the zero-mode is granted by translation invariance: given that $\rho(x)$ is a solution, a deformation of it given by $\rho(x+b)$, with constant b, is still a solution and this is

associated to the mode with $\omega_0 = 0$.
This is a general fact: consider any soliton solution $s(x)$ which, for some symmetry reason like translational invariance, does not come alone but is part of a family of solutions

$$s(x; \vec{b})$$

labeled by a set of coordinates $\vec{b}$.
These coordinates are called "collective coordinates" and each is associated to a mode ξ_0 of zero frequency, which in turn leads to a "free particle" hamiltonian, as in the above equation.

The quantization is now straight-forward, we have found a system which is the sum of harmonic oscillators and of a free particles, the states are:

$$|P_0\rangle, \text{ with } E = E_s + \frac{\hbar P_0^2}{2} + \delta E$$

$$|N_n\rangle, \text{ with } E = \hbar\omega_n \cdot N_n + E_s + \delta E$$

plus states made of combinations of the two,

$$|N_n\rangle |P_0\rangle, \ldots,$$

usual scalar excitations around the trivial vacuum.