S. Coleman
"Aspects of Symmetries"
Chapter 6

• Solitons •

The existence of particles associated to the quantum fluctuations of the fields around trivial (i.e., vanishing, or constant in space) configurations is by far the most important and phenomenological established manifestation of a QFT. More subtle effects, associated to field configurations of non-trivial type, are also important, even though their experimental effects are much more difficult to establish and to compare with observations. The first class of such effects we want to talk about is the existence of a new type of particle, the <u>solitons</u>, that emerge as quantum fluctuations of the fields around non-trivial, static, classical field configurations. What I call soliton in this lecture is exactly the same that Coleman calls "lumps" in his lecture. The word soliton, and not lump, is the one generically adopted in the literature.

Our final goal (to be achieved, however, only in the next Lecture) is the study of the quantum implications of the soliton. For the time being, however, let us study it purely at the classical level. In a classical field theory, a Soliton is defined as a "Regular non-dissipative solution of finite energy". With "solution", we mean that the soliton must be a solution of the classical equations of motions. By "non-dissipative", we mean that the energy density $\rho[\Phi_s(\vec{x},t)]$, where Φ_s is the soliton, does not go to zero (as a function of $\vec{x}$, i.e., at any $\vec{x}$) for $t \to \infty$. The soliton is by definition a field configuration that does not disappear with time. The most frequent case is when the soliton does not depend on time at all, the solition is indeed often defined to be a static field configuration, in the literature.

The simplest example of solitons is encountered in the theory of a real scalar in 1+1 dimensions:

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-U(\phi)$$

from which, by Legendre transform, we obtain the energy of a given field configuration

$$E=T+V=\int dx\left[\frac{1}{2}(\partial_0\phi)^2+\frac{1}{2}(\partial_1\phi)^2+U(\phi)\right]$$

The example we will consider are

$$U=\frac{\lambda}{2}\left(\phi^2-a^2\right)^2 \to \lambda\phi^4 \text{ theory}$$

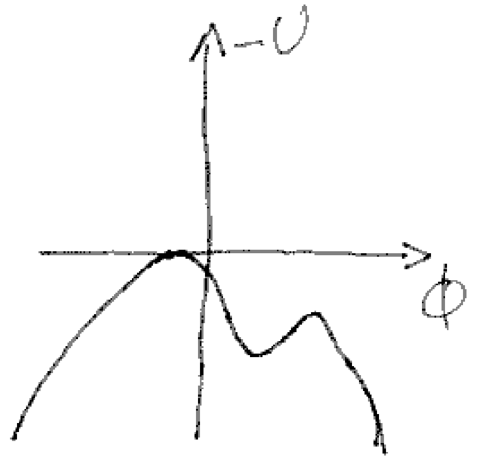
$$U=\frac{\alpha}{\beta^2}\left(1-\cos[\beta\phi]\right)\to \text{sine-Gordon model}$$

The problem can however be solved in general, a <u>static</u> solution of the Equations of Motion (EOM) is obtained by minimizing the energy:

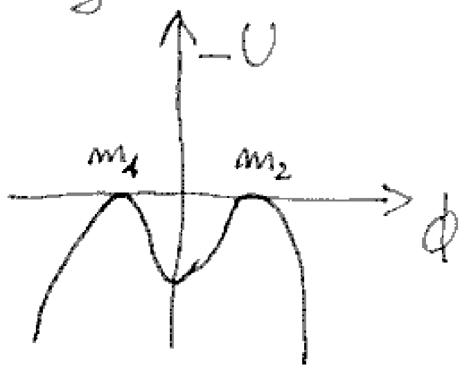
$$0=\delta V=\delta\int dx\left[\frac{1}{2}(\partial_1\phi)^2+U(\phi)\right]$$

where now $\phi=\phi(x)$. The analogy of the above system with the mechanical (classical) problem of a particle in 1d helps understanding.

You have of course to interpret $x \to t$, $\phi \to x$, and you find that the analog problem is the motion of a particle of unit mass in a potential equal to <u>minus</u> $U(\phi)$. If U is bounded from below, and shifted such as to vanish at the absolute minimum, the analog classical potential looks, in general, like:



In this case, however, we will not find solitons, the case we are interested in is when, as in the examples, there are at least two degenerate minima:



The point is of course that not only we need the soliton to be a solution of the EOM, we also need it to be of finite energy. Was only for being a solution we would have plenty of them, independently of the number of minima. But in order for them to have finite energy

$$E = \int dx \left[\frac{1}{2} (\partial_x \phi)^2 + U(\phi) \right] = \text{"finite"} \Rightarrow \begin{cases} \partial \phi \xrightarrow{x \to \pm\infty} 0 \\ \phi \xrightarrow{x \to \pm\infty} m_{1,2} \end{cases}$$

the point is that at infinity ($\pm\infty$) the field needs:

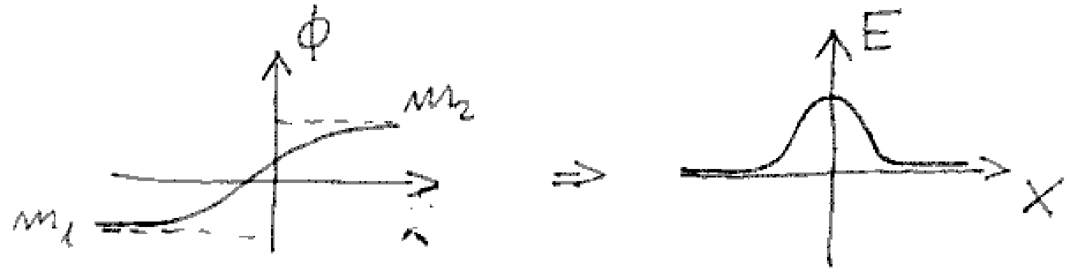
1) To approach a constant
2) to approach the (or one) absolute minimum

If at $x = +\infty$ and $x = -\infty$ we take our field to be at the same minimal point, it will just stand at that point for any x. This is just the constant ordinary vacuum solution (one for each minimum) that we usually employ for the quantization and gives rise to ordinary particles.

If instead we consider "boundary conditions"

$$\phi(-\infty)=m_1\,,\quad \phi(+\infty)=m_2\neq m_1$$

the solution is non-trivial. It correspond, in the mechanical analog, to the particle starting almost at rest for $t\to-\infty$, almost sitting at m_1. The particle then starts rolling down until it gets to m_2, where is asymptotically stops. A pictorial representation of this "motion" is:



don't get confused by the word "motion", the soliton is a static object, it connects the two vacua in space, not in time. Notice the shape of the energy density, the soliton is localized in space. Of course, we have no idea of where it is localized. Given that the theory is translational invariant we have unavoidably a family of solutions. If $f(x)$ is a solution, also $f(x-b)$

The explicit form of the solution is easy to obtain in the explicit examples we introduced before, but the derivation is left as an exercise. The results are

$$V=\frac{\lambda}{2}(\phi^2-a^2)^2 \longrightarrow f(x)=a\tanh(\mu x) \quad ; \mu=a\sqrt{\lambda}$$

$$E=\frac{4\mu^3}{3\lambda}$$

$$\Downarrow$$

this is the famous "Kink"

$$V=\frac{\alpha}{\beta^2}\left(1-\cos[\beta\phi]\right) \longrightarrow f(x)=\frac{4}{\beta}\arctan\left[e^{\sqrt{\alpha}x}\right]$$

$$E=\frac{8\sqrt{\alpha}}{\beta^2}$$

Notice that on top of the above solutions there are also the ones obtained by $x\to -x$, which correspond to "moving back" from the largest minimum to the smallest. These are called "anti-soliton", or "anti-Kink".

Already at the classical level, and even more at the quantum one, it is important to establish if the solution we have found is stable or not under small perturbations. If it is not, the soliton would never appear in "practical life" (provided you build a classical system governed by the previously-described theory) due to experimental noise. In quantum mechanics, where the fluctuations are unavoidable, the non-stability would make impossible to quantize the theory due to tachionic states. Instead, <u>the soliton is stable</u> (in our case):

$$\phi(x,t) = f(x) + \delta(x,t)$$

the evolution of the perturbation δ is governed, obviously, by the complete time-dependent EOM

$$\Box\phi + U'[\phi] = \Box\delta + U''[f]\cdot\delta = 0$$

because of time-translation invariance

$$\delta(x,t) = \sum_n a_n e^{i\omega_n t}\psi_n(x) + h.c.$$

$$-\frac{d^2\psi_n}{dx^2} + U''(f)\psi_n = \omega_n^2\psi_n$$

Stability means that the perturbation <u>remain small</u> in time. This is so if

$$\omega_n^2 \equiv \mathcal{E}_n \geq 0 \quad \forall n$$

also in this case, an analogy helps. The analogy with the Schrödinger equation with potential $V(x)=U''[\varphi(x)]$. In particular, it helps the theorem that "the eigenfunction with no nodes is the one of lowest energy". But we already know one solution, with zero energy:

$$\varphi(x+a) \cong \varphi(x) + a \underbrace{\frac{d\varphi}{dx}}_{\psi_0(x)}$$

but φ is monotonic, so that ψ_0 has no nodes, therefore it has minimal energy, $\omega_n^2 \geq 0$

We have checked explicitely the existence of solitons in the "1+1" scalar theory with degenerate minima. We might however have proven the existence of such a soliton with a different line of argument, which more easily generalizes to more complicated situations, i.e. by the use of Topology. Consider again the scalar theory, and remember that a soliton is a generic non-dissipative solution: it needs not to be static, the important point is that it does not disappear in the course of time. Now, since it also has to have finite energy (at any fixed time, energy is conserved), it must be such that at a given "initial" time $-T$:

$$\lim_{x\to\pm\infty}\phi(x,-T)\equiv\phi(\pm\infty,-T)=m_{1,2}, \quad (1)$$

where $m_{1,2}$ are the degenerate minima.
Let us denote, for clarity,

$$\phi_0(x),\ [\partial_0\phi_0](x),\ [\partial_0^2\phi_0](x),\ \ldots,$$

the initial-value data for the field and for its derivatives at $t=-T$. Normally, specifying the I derivative is enough, but this is irrelevant for the following argument.
Now, the condition in eq. (1) obliges the space of all possible initial values, $\phi_0(x)$, to split into four disconnected subsets:

ϕ_0

A: $\phi_0(+\infty)=m_1$, $\phi_0(-\infty)=m_1$	C: $\phi_0(+\infty)=m_1$, $\phi_0(-\infty)=m_2$
D: $\phi_0(+\infty)=m_2$, $\phi_0(-\infty)=m_1$	B: $\phi_0(+\infty)=m_2$, $\phi_0(+\infty)=m_2$

Now, the point is that the time evolution cannot make the solution exit the set it was in at $t=-T$. The "label of the set", A, B, C or D can be considered as a discrete quantum number. Given that it is conserved by time evolution, it is the simplest example of what is called a "topologically conserved" quantum number. This conservation does not depend at all on symmetries of the theory, and not even on some special detail of its time evolution. It is just that the finiteness of the energy implies that at any fixed t the solution approaches one of the minima at $x=\pm\infty$, then it must be constant in time.

$$\partial_0 \phi(\pm\infty, t) = 0$$

If the time evolution is smooth, therefore, the value at $\pm\infty$ stays constant from $-T$ until the end of the evolution. An equivalent way to state the same is that it is impossible to smoothly deform one initial value function

$\Phi_0(x)$ to any other one, $\Phi_0'(x)$, in a different class.

The existence of class $\mathbb{C}$ and D, where all the initial values $\Phi_0(x)$ are non trivial, implies the existence of the soliton. Any initial condition in C or D leads to a solution that cannot be changed into the trivial one (which lines in A, B) by time evolution. Remember however that we are interested in static solitons (I would not know how to quantize a non-static one, and what to make with it), that ones we are not guaranteed that they exist by the topology argument. One has to really inspect the EOM.

If this concept is clear in this simple case, most of the seeming complication of the arguments based on topology rapidly disappears. As a first example, consider the Vortex, or the "flux lines" in superconductor's theory. These are solitons that appear in 1+2 dimensions in the U(1) gauge theory of one complex scalar. This is just the abelian Higgs model in 1+2:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi +$$
$$- \frac{\lambda}{2} \left(\phi^\dagger \phi - a^2 \right)^2$$

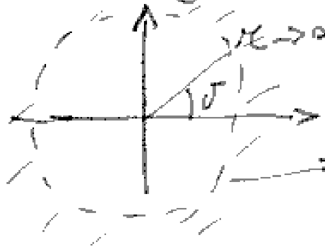
$D_\mu = \partial_\mu + i e A_\mu$, with the local $U(1)$ gauge group acting as

$$\phi \longrightarrow e^{i\omega(x)} \phi$$
$$A_\mu \longrightarrow A_\mu - \partial_\mu \omega / e$$

As you know very well, the vacuum of the theory is given by

$$\phi = a .$$

The one above is, up to gauge transformations, the only zero-energy field configuration. Consider now, as before, a fixed time t, we have to ensure that the energy is finite, and for this we ask that the energy density vanishes at $r \to \infty$:



the field here must approach the trivial vacuum

Denoting generically our fields as Ψ:

$$\Psi\big|_{r=\infty} = \left(\Psi_{VAC}\right)^{(\hat{g})}$$

where "$\hat{g}$" is an element of the <u>local group</u>. The transformation "$\hat{g}$" can still depend on the <u>direction</u> from which we approach infinity. In our case, this is to say that it can depend on θ. In general, for "d" space-time dimensions, "$\hat{g}$" is a function on the "sphere at infinity":

$$\hat{g} : S_{d-1} \longrightarrow G$$

where G is the gauge group ($U(1)$, in our case). Denote as $\vec{\xi}$ the coordinates of S_{d-1},

$$\hat{g} = \hat{g}(\vec{\xi}),$$

it happens, for some choice of G and of d, that the maps like $\hat{g}$ separate into different topological classes, similarly to the ones found for the scalars in 1+1. Those are called "homotopy classes". Each class is defined as the set of maps that can be continuously deformed into each other. It is an equivalence class, where two maps are equivalent if and only if they are deformable into each other:

$$\hat{g}_1(\vec{\xi}) = \hat{G}(\xi; -T)$$
$$\hat{g}_2(\vec{\xi}) = \hat{G}(\xi; +T)$$

If such a map does not exist, the maps are "topologically inequivalent".

For the case at hand, we have that each class C_n is represented by

$$\hat{g}_n(\sigma) = e^{in\sigma}$$

with $n \in \mathbb{Z}$. This means that the homotopy classes form a "$\mathbb{Z}$-group", in the sense that their multiplication obeys the same algebra as the addition of integers does. Mathematically, this we state as

$$\Pi_1(U(1)) = \mathbb{Z}$$

Where Π_1 is called "the first homotopy group"

In more dimensions, you have to look at

$$\pi_{d-1}(G)$$

If it is non-trivial, we have topologically disconnected sets.

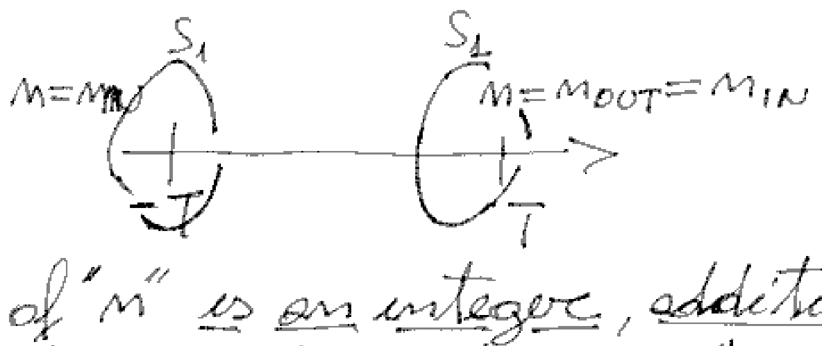
Let us go back to our vortex, suppose you impose, as boundary condition at infinity:

$$\phi|_{\infty} = e^{in\theta} * a$$

$$\vec{A}|_{\infty} = -n\vec{\theta}/e$$

$$A_0|_{\infty} = 0$$

i.e., suppose that you impose your field to reduce to the trivial vacuum rotate with a topologically non-trivial gauge transformation. Time evolution cannot change the value of n:



The value of "n" is an integer, additive conserved charge. It is not a Noether charge, however, it is a topological charge!

To make this more concrete, let us work out the solution of the vortex more explicitely. We consider static configurations, with on top of this, vanishing A_0 field. The energy for such configurations is

$$E = \int d^2x \left[\frac{1}{4} F_{ij} F_{ij} + (D_i \phi)^+ (D_i \phi) + \frac{\lambda}{2} (\phi^+ \phi - e^2)^2 \right]$$

where: $D_i = \partial_i + ie A_i$; $i = 1, 2$. It is convenient to make an ansatz, remember however that a "good" ansatz is always based on symmetries. Consider the symmetry of rotation in space:

$$\phi(\vec{x}) \longrightarrow \phi(R \cdot \vec{x})$$
$$A_i(\vec{x}) \longrightarrow (R^{-1})_{ij} A_j (R \cdot \vec{x})$$

and the one of $U(1)$ global phase transform:

$$\phi(\vec{x}) \longrightarrow e^{i\omega} \phi(x)$$
$$A_i(\vec{x}) \longrightarrow A_i(\vec{x})$$

Imposing the ansatz to be invariant under any of these $U(1)=SO(2)$ symmetries would not lead to any solution, we instead impose the solution to be invariant under one combination of the two symmetries, which we call cylindrical symmetry:

$$\bar{\Phi}(\vec{x}) = e^{in\omega}\,\bar{\Phi}(R^{\omega}\cdot\vec{x})$$
$$\bar{A}_i(\vec{x}) = (R^{\omega-1})_{ij}\,\bar{A}_j(R^{\omega}\cdot\vec{x})$$

where R^{ω} is the rotation of angle ω and n is a positive or negative integer. For each "n", we have a different ansatz, obtained by solving the condition above. This is given by:

$$\bar{A}_i(\vec{x}) \equiv \frac{x_i}{r^2}\tilde{A}(r) - \varepsilon_{ij}\frac{x_j}{r^2}A(r)$$
$$\bar{\Phi}(\vec{x}) \equiv e^{in\theta} f(r)$$

where

$$\theta = \arctan[x_2/x_1]$$

is the polar angle, which shifts under the rotation. Of course, $r=\sqrt{x_1^2+x_2^2}$

Thank to the Ansatz, our solution now depends only on three unknown functions of a single variable, that can be determined by solving the EOM, but with which boundary conditions? Remember that we have to impose the fields at infinity to reduce to the trivial vacuum rotated by $\hat{g}(\theta)$:

$$\begin{pmatrix} \bar{\phi} \\ \bar{A} \end{pmatrix}\bigg|_{r\to\infty} = \begin{pmatrix} a \\ 0 \end{pmatrix}^{\hat{g}} \quad ; \quad \hat{g} = e^{in\theta}$$

$$\Rightarrow \quad \bar{\phi}_{|\infty} = a e^{in\theta} \iff f_{|\infty} = a$$

$$\bar{A}_{i|\infty} = -n\partial_i\theta = -n\varepsilon_{ij}\frac{x_j}{r^2} \iff \begin{matrix} \tilde{A}_{|\infty} = 0 \\ A_{|\infty} = \frac{n}{e} \end{matrix}$$

The above equations ensure that the energy is finite for $r\to\infty$, we must however still worry on what happens for $r\to 0$. Given that we are working in polar coordinates, we are not authomatically guaranteed that our fields are <u>regular</u> <u>and</u> single-<u>valued</u> at $r\to 0$.

For that, we must ask

$$f(0) = A(0) = \tilde{A}(0) = 0$$

with A and $\tilde{A}$ going faster than r.

At this point, we just have to fix the gauge, we choose:

$$\partial_r A_r = 0 \iff \boxed{\tilde{A} = 0}$$

From now on, it is straightforward to obtain the EOM for $f(r)$ and $A(r)$, by plugging the Ansatz in the energy and deriving the EOM by varying the energy with respect to f and A.
You are asked to complete this program in the exercise.

The discussion has not yet touched the most important (in particle physics) applications of the theory of solitons, but now at least you have the instruments to understand what they are without entering into details.

Monopole: In $d=4$, consider an $SO(3)$ gauge theory, with scalars ϕ_i in the triplet. The scalar potential

$$V = \lambda(\phi_i \phi_i - v^2)^2$$

breaks spontaneously $SO(3)$ to $SO(2)$. Let us now consider, as usual, the allowed boundary conditions that, at any fixed time, we can impose at the 3-space infinity. In order for the energy to be finite, we must have

$$(\phi_i \phi_i)_{|\infty} = v^2$$

while $\phi_{i|\infty}$ itself can still depend on the coordinates θ and ϕ of the S_2 at space infinity. Given the above condition, $\phi_{i|\infty}$ is a point on an S_2, in the space of fields, of course, not in the physical space. It is then a map

$$\phi_{i|\infty} : S_2 \longrightarrow S_2$$

These maps are classified by

$$\pi_2(S_2) = Z$$

therefore, non-trivial $\phi_{k|\infty}$ exist, which the time evolution cannot deform to trivial, these are the <u>monopoles</u>.

<u>Skyrmions</u>: Here we consider the chiral $SU(2)$ σ-model which describes the pions. Consider, at each given time t, the Goldstone matrix

$$U(\vec{x}; t) \in SU(2)$$

For it having finite energy, always at fixed time t, we must have that

$$U(\vec{\infty}; t) = \overline{U}(t)$$

where $\overline{U}$, given that now we have no gauge invariance, <u>cannot depend on the point on the S_2 at infinity</u>, if not the energy would diverge because of its derivative terms.

But then, $U(\vec{x};t)$ is single-valued, at each given t, also at infinity. We can then regard it as a map not on $\mathbb{R}_3$ of 3-space, but on S_3, i.e. $\mathbb{R}_3$ plus the point at infinity:

$$\forall t \; : \; U(\vec{x};t) : S_3 \longrightarrow S_3$$

but given that

$$\pi_3(S_3) = \mathbb{Z}$$

topologically non-trivial configurations exist, which cannot be deformed to each other by the time evolution. As in other cases we already encountered, we have a conserved topological charge that guarantees the existence of the soliton. A static soliton or not, of course, depends on the EOM, for the Skyrmion, the existence of a static solution is an issue, but this is another story. The additive integer charge of the Skyrmion was interpreted by Skyrme as the Baryonic Charge. The proposal of Skyrme was that the Skyrmions, i.e. solitons in the theory of pions, describe the physical baryons.

Something more on Monopoles and Skyrmions can be found on Weinberg Chapter 23