Weinberg 22.3, 22.4

Anomaly Cancellation

One important application of the calculation of the Anomaly which we performed in the last lecture is that we can use it to check that the Anomaly cancels in the theories (typically, gauge theories) we use to describe Nature. This must be so, for internal consistency. Let us start, before going to the Standard Model of Weak and Strong interactions, by giving few simple examples of cases in which the Anomaly cancellation is simple, and automatic.

First, the anomaly cancels always if we have a vector-like theory, i.e. if we have a theory where the Left and Right-handed fermions transform in exactly the same way under the group. For this reason, we do not need to worry about anomalies in QED or in QCD. Better said, this is why all the correlator with electromagnetic and QCD currents <u>only</u> are free of Anomalies. There might be, and <u>indeed</u> <u>there</u> <u>are</u>, mixed anomalies with 2

QCD and one global (chiral) current. It is very simple to see how this works. The generators are

$$T^\alpha = \begin{pmatrix} t^\alpha & 0 \\ 0 & -(t^\alpha)^* = -(t^\alpha)^t \end{pmatrix}$$

$$D = \frac{1}{2} \mathrm{tr}\left[\begin{pmatrix} \{t^\alpha, t^\beta\} t^\gamma & 0 \\ 0 & -(t^\gamma\{t^\alpha, t^b\})^t \end{pmatrix}\right] = 0$$

to appreciate the difference with the <u>chiral</u> case, in which instead the Anomaly appears, consider the simple case of <u>one</u> <u>Dirac</u> with $m=0$, this theory is endowed with two $U(1)_L \times U(1)_R$ global group

$$\psi_L \to e^{i\alpha_L}\psi_L \quad ; \quad \psi_R \to e^{i\alpha_R}\psi_R$$

$$\Rightarrow T^L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad T^R = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

We can go to the Vector-Axial basis and compute the anomaly associated with

$$\langle VVV \rangle : \alpha \, \mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

or instead

$$\langle AVV\rangle : \propto \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$$

this tells us that the anomaly is present in this case, but that we can always eliminate from the vector current, because $\langle VVV\rangle = 0$, consistently with what we saw before with the case of vector-like symmetries. We cannot instead preserve the axial since $\langle AAA\rangle \neq 0$

Another interesting case is when the theory is chiral, let us say that we have only L-handed fields, but the group is real. In this case we have:

$$\begin{aligned} D_{\alpha\beta\gamma} &= \mathrm{tr}\left[\{t_\alpha, t_\beta\} t_\gamma\right] = \\ &= \mathrm{tr}\left[(t_\gamma)^t \{(t_\alpha)^t, (t_\beta)^t\}\right] = \\ &= -\mathrm{tr}\left[t_\gamma \{t_\alpha, t_\beta\}\right] \end{aligned}$$

where we used the fact that for a real group t_α's are antisymmetric. This also works for pseudo-real groups such as $SU(2)$. In that case you can see this explicitely by remembering that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$

Finally, let us show that, as promised in the previous lecture, all the fermions that are allowed to have a mass by the symmetry do not contribute to the anomaly. To see this, start from the mass-term

$$m_{ij}\,\bar{\psi}_R^{\,i}\,\psi_L^{\,j}$$

For this being invariant, as we assume, we must have:

$$+(t_R^*)_{i'i}\, m_{i'j} = m_{ij'}(t_L)_{j'j}$$

$$\Downarrow$$

$$t_R\, m = m\, t_L$$

If we define:

$$M = \begin{pmatrix} 0 & m^t \\ m & 0 \end{pmatrix}$$

the above condition becomes:

$$MT = -T^t M$$

$$\begin{pmatrix} 0 & -m^t t_R^t \\ m t_L & 0 \end{pmatrix} = \begin{pmatrix} 0 & -t_L^t m^t \\ t_R m & 0 \end{pmatrix}$$

From here on, the proof presented on the Weinberg follows straightforwardly, you are invited to read and reproduce it.

Let us now consider a concrete example of a gauge theory, of great phenomenological importance, the Standard Model of EW and strong Interaction. Given that the Weak interactions are <u>chiral</u> in Nature, and this is why they <u>break Parity</u>, the SM is a <u>chiral gauge theory</u>, anomaly cancellation is therefore a non-trivial constraint that needs to be satisfied for internal consistency. Let us list the field content of the SM, in the chiral basis of L-handed fields

	$SU(3)$	$SU(2)$	$U_Y(1)$
$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\underset{\sim}{3}$	$\underset{\sim}{2}$	$1/6$
$(u_R)^c$	$\underset{\sim}{\bar{3}}$	$\underset{\sim}{1}$	$-2/3$
$(d_R)^c$	$\underset{\sim}{\bar{3}}$	$\underset{\sim}{1}$	$1/3$
$E_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}$	$\underset{\sim}{1}$	$\underset{\sim}{2}$	$-1/2$
$(e_R)^c$	1	$\underset{\sim}{1}$	1

where, remember, $SU(3)$ is the QCD group, $SU(2)\times U(1)_Y$ the EW one, with the electric charge Q

$$Q = T^3 + Y$$

We want to check the cancellation of the D symbol

$$D^{\alpha\beta\gamma} = \frac{1}{2} \mathrm{tr}\left[\{T^\alpha, T^\beta\} T^\gamma\right]$$

for all possible choices of $T^{\alpha,\beta,\gamma}$ within the gauged group $SU(3)\times SU(2)\times U(1)_Y$. Notice that D is an <u>invariant tensor</u>:

$$D \longrightarrow D = \frac{1}{2} \mathrm{tr}\left[\{g T^\alpha g^+, g T^\beta g^+\} \cdot \times g T^\gamma g^+\right]$$

so that it could be different from zero only for a combination of generators such that one <u>singlet</u> of $SU(3)\times SU(2)\times U(1)_Y$ can be formed out of them. For example, one $SU(3)$ and two $U(1)_Y$ currents cannot have a non-vanishing D-symbol because an invariant cannot be formed, you can also check this explicitely.

the combinations it is worth worrying about are:

<u>$SU(3)-SU(3)-SU(3)$</u>:

D vanishes because the quarks are a <u>vector-like</u> representation of $SU(3)$

<u>$SU(3)-SU(3)-U(1)_Y$</u>:

The $SU(3)$ generators take the form

$$\begin{array}{c|c} q & \\ & \\ & \\ & \\ \text{lept} & \\ & \\ & \\ & \end{array}\begin{pmatrix} \lambda^a & & & & & & & \\ & \lambda^a & & & & & & \\ & & -\lambda^{a*} & & & & & \\ & & & -\lambda^{a*} & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

the $U(1)_Y$ is

$$\begin{pmatrix} 1/6 & & & & & & \\ & 1/6 & & & & & \\ & & -2/3 & & & & \\ & & & 1/3 & & & \\ & & & & -1/2 & & \\ & & & & & -1/2 & \\ & & & & & & 1 \end{pmatrix}$$

$$D = D^{abY} \propto \frac{1}{2}\underbrace{\mathrm{tr}\left[\{\lambda^a,\lambda^b\}\right]}_{=\delta^{ab}} \times \underbrace{\left[\underbrace{\frac{1}{6}\times 3}_{u_L} + \underbrace{\frac{1}{6}\times 3}_{d_L} - \underbrace{\frac{2}{3}\times 3}_{u_R} + \underbrace{\frac{1}{3}\times 3}_{d_R}\right]}_{=0\,!}$$

Notice that D is proportional to δ^{ab} because it has to be invariant under $SU(3)$.

<u>$SU(2)-SU(2)-SU(2)$:</u>

Here $D=0$ because $SU(2)$ is anomaly-free

<u>$SU(2)-SU(2)-U(1)_Y$:</u>

The generators now have the form

$$\underbrace{\begin{pmatrix} \overbrace{\sigma\ \sigma\ \sigma}^{q_L} & & & & \\ & \overbrace{0\ 0\ 0}^{u_R} & & & \\ & & \overbrace{0\ 0\ 0}^{d_R} & & \\ & & & \overbrace{\sigma}^{E_L} & \\ & & & & \overbrace{0}^{e_R^c} \end{pmatrix}}_{SU(2)} \qquad \underbrace{\begin{pmatrix} \overbrace{1/6\ 1/6\ 1/6}^{q_L} & & & \\ & \ddots & & \\ & & \overbrace{-1/2}^{E_L} & \\ & & & \ddots \end{pmatrix}}_{U(1)_Y}$$

$$D = \frac{1}{2} \operatorname{tr}[\sigma^a \sigma^b] \left[3 \times \frac{1}{6} - \frac{1}{2} \right] = \underline{0} \ !$$

<u>$U(1)_Y - U(1)_Y - U(1)_Y$:</u>

The trace of Y^3 is:

$$\operatorname{tr}[Y^3] = \left(\frac{1}{6}\right)^3 \times 3 \times 2 - \left(\frac{2}{3}\right)^3 \times 3 + \left(\frac{1}{3}\right)^3 \times 3 + \\ - \left(\frac{1}{2}\right)^3 \times 2 + (1)^3 = \left(\frac{1}{6}\right)^2 - \frac{8}{9} + \frac{1}{9} - \frac{1}{4} + 1$$

$$= \frac{1}{4 \cdot 9} - \frac{28}{4 \cdot 9} - \frac{9}{4 \cdot 9} + \frac{36}{4 \cdot 9} = 0 \ !$$

This condition is also non-trivially satisfied, we have then shown that all the gauge anomalies cancel in the SM. Actually, there is a last anomaly we should worry about, the mixed gauge-graviton-graviton. This comes from correlators $\langle J T T \rangle$, where T is the energy-momentum tensor. Given that gravity is universal, it is not so surprising that this anomaly is proportional to

$$\mathrm{tr}[T_a]$$

To cancel also this latter anomaly one needs all generators being traceless. This can be checked for the Hypercharge.

The condition of anomaly cancellation might have been imposed, and used to determine the correct hypercharge assignements.

Until now we have discussed anomalies of gauged symmetries, that have to cancel. Let us now discuss global currents, these can be anomalous without any problem. Even if the anomaly is with some gauged current, it doesn't matter because we will always be allowed to put the anomaly on the global leg, using a gauge-invariant regulator.

First consider Baryon Number, an important accidental global symmetry of the SM. It is accidental because at the renormalizable level, with the charge assignements of the fermions and of the Higgs doublet (in the 2, $Y = 1/2$). Up to non-renormalizable terms, that you could put in the Lagrangian suppressed by an arbitrarily high scale, B is therefore a symmetry of the SM at the classical level. At the quantum level, however, consider the $U(1)_B - SU(2) - SU(2)$ anomaly, this gives

$$\mathrm{tr}[\sigma^a\sigma^b] \times \left[\frac{1}{3} \times 3\right] \neq 0$$

having assigned 1/3 B-charge to the quarks. Therefore, there exists a mixed $U(1)_B - SU(2) - SU(2)$ anomaly, the conservation of the Baryon number is violated in the SM!

Another interesting accidental symmetry is Lepton Number. The $U(1)_L - SU(2) - SU(2)$ anomaly reads

$$\mathrm{tr}[\sigma^a\sigma^b] \times \left[-1\right] \neq 0$$

having given $L = -1$ to $\binom{\nu}{e}$. The lepton number is also violated, but notice that

B − L is Anomaly-free