FROM LECTURES OF ADAM SCHWIMMER

The Chiral Anomoly in 2d

The problem of anomolies has no counterpart in classical physics. The Anomaly can be defined as an obstruction which forbids to uplift a certain symmetry of a classical Field Theory to a complete symmetry of its Quantum (Quantized) version. Poorly speaking, the anomoly is a breaking of a "Classical" symmetry which is induced by Quantum Effects.

Before going to details, and explaining how this can happen, let us make clear what the implications of the presence of an anomoly in a theory can be. The anomoly will result, at the end of the day, into the violation of some identity, which would be implied by the symmetry under consideration. For instance, if it happened that the U(1) symmetry of QED was violated by Anomolies (remember, it does not happen), you will encounter violations of the Ward Identities you have studied in QFT-1. The implications of such a violation are very different, depending on whether the anomaalous symmetry was a global or a local one. If it is global, such a violation is perfectly admissible, it simply leads to a modification of the physical laws that could be inferred from the identity (i.e., ultimately, from the symmetry). It is not even that the implications of the symmetry are "lost" due to the anomaly. The anomaly leads to perfectly predictable modifications of the Ward identities. In QCD, one such modification (that corresponds to an anomaly of the global QCD (chiral) group) takes place, can be computed, and leads to the sharp prediction of the $\pi^0 \to \gamma\gamma$ decay rate. Theories with Global Anomalies are therefore perfectly allowed. The case of gauge symmetries is completely different: a local gauge symmetry cannot be anomalous. The reason is that the gauge symmetry is not a symmetry, it is a redundancy of the parametrization of the physical states in terms of fields. This is to say that the A_μ gauge fields contain more degrees of freedom than the two physical polarizations, and one of the other degrees of freedom is a "ghost", i.e. a state of negative norm. You know that a theory with such a particle is <u>inconsistent</u>, and that the role of the gauge-invariance of the Lagrangian, which implies the Ward identities, is to make these unphysical states <u>decouple</u>. With an anomaly, the ghosts <u>would not decouple</u> any longer, and the theory would become <u>inconsistent</u>. When formulating a gauge theory, therefore, we should carefully <u>check that the anomalies cancel</u>. We will verify that this anomaly cancellation takes place in the Standard Model of Electro-Weak and Strong interactions.

The anomaly takes place because of the regulator (for instance, the cutoff Λ or the ε of $d=4-\varepsilon$ in dimensional regularization) that is unavoidably introduced as an intermediate step of the calculations. In order to derive <u>in a robust way</u> all the implications of the symmetry, it should happen that <u>the regulated theory is also</u> invariant under the symmetry. If instead the regulator <u>is not invariant</u>, the implications of the symmetry are typically violated, and they could not even be restored when the regulator is removed (i.e., for $\Lambda \to \infty$ or $\varepsilon \to 0$). This is why it is so important to work with a regulator that respects all the symmetries of the theory. It might happen, however, that such a regulator does not exist, and this is when we have an Anomaly: the Anomaly is a "clash" among different symmetries and it takes place when all the different symetries of the theory cannot be imposed simultaneously to hold in the regulated theory. We will see examples of this clashes.

The simplest possible Anomaly is the chiral Anomaly in 2 dimensions (1 time + 1 space), to study it let us first of all define the spinors in 2d. The gamma matrices γ^μ, with $\mu = \{0, 1\}$, must satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad ; \eta = \mathrm{diag}(+,-)$$

and the smallest matrices for which these relations can be enforced are 2×2, with

$$\gamma^0 \equiv \sigma_2 \; ; \quad \gamma^1 \equiv -i\sigma_1$$

Also in 2d it exists the analog of γ^5, that we call γ^3, which is defined as

$$\gamma^3 = -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

The γ^3 can be used, as γ^5 in 4d, to define the chiral representations of the Lorentz group, this obviously could be done in any even number of space-time dimensions. The point is that the spinors ψ are 2-components vectors, on which the Lorentz Group's algebra acts with generators

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Given that γ^3 commutes with Σ, if ψ forms a representation,

$$\psi_{\substack{L\\R}} \equiv \frac{1 \pm \gamma^3}{2}\psi$$

form a representation as well. A 2-spinor ψ can therefore always decomposed as:

$$\psi = \psi_L + \psi_R = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}$$

The chiral spinors $\psi_{L,R}$ are the truly irreducible representation of Lorentz in 2d (also called $SO(1,1)$). Notice that Lorentz in 2d is an abelian group (the only non-trivial generator is Σ^{01}), but not compact (the generator is not hermitian).

In 2d we can define parity as well, which acts on the spinor fields as

$$P\psi(t,x)P^{+} = \gamma^0\psi(t,-x)$$

so that P exchanges Left- and Right-handed:

$$P\psi_L(t,x)P^{+} = \frac{1+\gamma^3}{2}\gamma^0\psi(t,-x) =$$
$$= \gamma^0\psi_R(t,-x)$$

Let us consider the simplest possible theory, which contains two Weyl fermions $\psi_{L,R}$, it is free and massless:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\bar{\psi}_L \not{\partial}\psi_L + i\bar{\psi}_R\not{\partial}\psi_R$$

This theory, like QCD, is invariant under independent $U(1)_L \times U(1)_R$ transformations:

$$\psi_L \to e^{i\alpha_L}\psi_L, \qquad \psi_R \to e^{i\alpha_R}\psi_R$$

The associated Noether currents are

$$L_\mu = \bar{\psi}_L \gamma_\mu \psi_L = \bar{\psi}\gamma_\mu \frac{1+\gamma^3}{2}\psi$$

$$R_\mu = \bar{\psi}\gamma_\mu \frac{1-\gamma^3}{2}\psi$$

$$V_\mu \equiv L_\mu + R_\mu = \bar{\psi}\gamma_\mu \psi$$

$$A_\mu \equiv L_\mu - R_\mu = \bar{\psi}\gamma_\mu \gamma^3 \psi$$

Let us derive the "naive" implications of these symmetries, using the path integral method, these implications are the "Ward Identities" of our symmetries, and will be violated by the anomaly. The method which follows, when it is not violated by an anomaly, is very useful to derive the implications of a symmetry, and it is easily generalized, so keep it in mind. We start by noticing that the currents satisfy, by definition, the relation

$$\delta S = \int dx\, \partial_\mu J^\mu \cdot \alpha(x)$$

where the variation of S comes from an x-dependent transformation in the global symmetry group. For instance, under a vector transformation ($\alpha_L = \alpha_R = \alpha$):

$$\delta_\alpha S = \int d^4x \left[\bar{\psi}^{(\alpha)} i \not{D} \psi^{(\alpha)} \right] - S =$$

$$= \int d^4x \, (-) \partial_\mu \alpha \, \bar{\psi} \gamma^\mu \psi$$

Consider the following identity

$$\int D\psi D\bar{\psi} \; A_\mu(x) \, e^{i S[\psi]} = \int D\psi^{(\alpha)} D\bar{\psi}^{(\alpha)} A_\mu^{(\alpha)}(x) \times$$
$$\times e^{i S[\psi^{(\alpha)}]}$$

this is just a relabeling of the integration variable, so it is definitely true. But use now:

1) $A_\mu^{(\alpha)} = \bar{\psi}^{(\alpha)} \gamma_\mu \gamma^3 \psi^{(\alpha)} = A_\mu$

2) $S[\psi^{(\alpha)}] = S[\psi] + \int \alpha \partial_\mu V^\mu$

3) $D\psi^{(\alpha)} D\bar{\psi}^{\alpha} = D\psi D\bar{\psi}$

and expand for small α, we get:

$$\int D\psi D\bar{\psi} \; A_\mu(x) \times i \int dy \, \alpha(y) \partial_\mu V^\mu(y) = 0$$

This implies that by defining

$$G_{\mu\nu}(x-y) = \langle 0|T(A_\mu(x)V_\nu(y))|0\rangle,$$

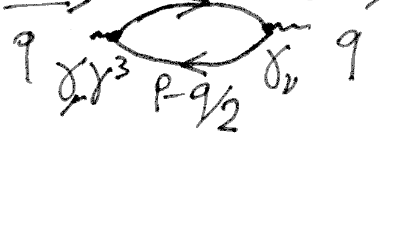
we have that $\partial^\nu G_{\mu\nu} = 0$, as a result of the <u>vector</u> symmetry.
Similarly, we might have proven that

$$\partial^\mu G_{\mu\nu} = 0$$

as a consequence of the <u>axial</u> symmetry. The above equation <u>will fail</u> due to the anomaly. In the above derivation, it is the step 3) that fails. We will not discuss this, but the anomaly originates from the non-invariance of the fermionic measure in the path integral. It is also in this case a problem with the regulator. The measure as well (as the loop integrals) needs to be defined (i.e., regulated) and it might be impossible to preserve all the symmetries in this process of regularization.

By going to Fourier space, we can compute this 2-point Green functions by Feynman diagrams:

$$G_{\mu\nu}|_{FT} \big/ (2\pi)^2\delta(q_1-q_2) \equiv \Pi_{\mu\nu}(q) =$$



Pay attention that we are computing in the free theory. The vertex "α" just represents the bilinear operator whose correlators we are willing to compute, it does not correspond to an interaction in the Lagrangian. This is another important point: the anomaly shows up in the free theory. Moreover, at least for the anomalies we will encounter in this course, the anomaly is not affected by the interactions. It can be computed in the free theory, and it is exact. This is how the QCD anomaly can be computed and used to predict the $\pi \to \gamma\gamma$ width, in spite of QCD being strongly-coupled.

Given that the propagator is just the usual

$$\longrightarrow \; = \frac{i \not{p}}{p^2 + i\varepsilon}, \quad \text{we find}$$

$$\Pi_{\mu\nu}(q) = + \frac{1}{(2\pi)^2} \int d^2 p \; \frac{\mathrm{tr}\left[\gamma^3 \gamma_\mu \left(\not{p} + \frac{\not{q}}{2}\right) \gamma_\nu \left(\not{p} - \frac{\not{q}}{2}\right)\right]}{\left[\left(p + q/2\right)^2 + i\varepsilon\right]\left[\left(p - q/2\right)^2 + i\varepsilon\right]}$$

one minus from the fermionic loop

and the Ward Identities that we would like to check are:

$$q^\mu \Pi_{\mu\nu} \overset{?}{\doteq} 0 \qquad \text{(axial conservation)}$$

$$q^\nu \Pi_{\mu\nu} \overset{?}{\doteq} 0 \qquad \text{(vector conservation)}$$

Notice that by naive manipulations one would get that both the identities are respected:

$$q^\mu \Pi_{\mu\nu} \propto \int d^2p \, \mathrm{tr}\left[\gamma^3 \not{q} \frac{1}{\not{p}+\not{q}/2} \gamma_\nu \frac{1}{\not{p}-\not{q}/2}\right] =$$

$$= \int d^2p \, \mathrm{tr}\left[\gamma^3 \cancel{(\not{p}+\not{q}/2)} \frac{1}{\cancel{\not{p}+\not{q}/2}} \gamma_\nu \frac{1}{\not{p}-\not{q}/2}\right]$$

$$- \int d^2p \, \mathrm{tr}\left[\gamma^3 (\not{p}-\not{q}/2) \frac{1}{\not{p}+\not{q}/2} \gamma_\nu \frac{1}{\not{p}-\not{q}/2}\right]$$

$$= \int d^2p \, \mathrm{tr}\left[\gamma^3 \gamma_\nu \frac{1}{\not{p}-\not{q}/2}\right] +$$

$$- \int d^2p \, \mathrm{tr}\left[\gamma^3 \gamma_\nu \frac{1}{\not{p}+\not{q}/2}\right] \overset{?}{\doteq} 0$$

this vanishes if I can make <u>shifts</u> in the integration variable p. In a regulated theory we might not be allowed of doing this. Notice that our diagram needs a regulator, it is logarithmic divergent. This is always the case, the anomaly arises in superficially divergent diagrams.

To compute the diagram, introduce first of all a Pauli-Villars regulator. That is, let us modify our theory by adding a massive fermion field Ψ_{PV} :

$$\mathcal{L} = \bar{\Psi} i \not{\partial} \Psi + \bar{\Psi}_{PV} i \not{\partial} \Psi_{PV} - M \bar{\Psi}_{PV} \Psi_{PV}$$

Only, Ψ_{PV} is quantized as a boson. This makes that it will systematically contribute with opposite sign than what Ψ does. In this way the divergences will cancel. The regulated versions of the V and A currents are

$$V_\mu^{reg} = V_\mu + V_\mu^{PV} \, ; \; A_\mu^{reg} = A_\mu + A_\mu^{PV}$$

where $V_\mu^{PV} = \bar{\Psi}_{PV} \gamma_\mu \Psi_{PV}$ and similarly for A_μ^{PV}.

Notice that the vector symmetry is respected by the regulator. The Ward identity which is associated to vector, therefore, cannot be violated with this regulator. The axial one, instead, is broken by the mass term!

A last step is needed in order for the calculation to be possible: we add a regulating mass m for the Ψ field as well. This is needed to avoid IR divergences, we will eventually get rid of m by taking the $m \to 0$ limit.

Let us finally perform the calculation: notice that

$$\gamma^3 \gamma_\mu = \varepsilon_{\mu\nu} \gamma^\nu \quad ; \quad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as can be checked explicitely. We therefore can write:

$$A_\mu = -\varepsilon_{\mu\nu} V^\nu \Rightarrow \Pi_{\mu\nu} \equiv \Pi^{AV}_{\mu\nu} = \varepsilon_{\mu\rho} \Pi^{VV\rho}{}_\nu$$

$$\Pi^{VV}_{\mu\nu} = \frac{1}{(2\pi)^2} \int d^2 P \, \frac{tr[\gamma_\mu (P + q/2 + m) \gamma_\nu (P - q/2 + m)]}{[(P + q/2)^2 - m^2 + i\varepsilon][(P - q/2)^2 - m^2 + i\varepsilon]}$$

In the PV regularization:

$$\Pi^{VV, reg}_{\mu\nu} = [\Pi^{VV}_{\mu\nu}(m) - \Pi^{VV}_{\mu\nu}(M_{PV})] \; ; \text{ for } \begin{matrix} m \to 0 \\ M_{PV} \to \infty \end{matrix}$$

Now the integral is finite, because we have made one subtraction, on the <u>integrand</u> of the difference among the "ψ" and "ψ_{PV}" contributions we can now make any manipulation, because the integral is now finite. Remember the crucial identities, that could be shown in 2d similarly to what one does in 4:

$$\mathrm{tr}[\gamma^\mu \gamma^\nu] = 2\eta^{\mu\nu} \quad ; \quad \mathrm{tr}[\gamma^\mu] = \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0$$

$$\Rightarrow \mathrm{tr}\left[\gamma_\mu (\not{P} + \not{q}/2 + m)\gamma_\nu(\not{P} - \not{q}/2 + m)\right] =$$

$$= \mathrm{tr}\left[\gamma_\mu(\not{P} + \frac{\not{q}}{2})\gamma_\nu(\not{P} - \not{q}/2)\right] + 2m^2\eta_{\mu\nu}$$

Also, as you will show as an exercise

$$\mathrm{tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma] = 2\left[\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right]$$

$\Rightarrow$

$$\mathrm{tr}[\,\cdots] = 2\Big[(P + q/2)_\mu (P - q/2)_\nu + (P - q/2)_\mu (P + q/2)_\nu + \eta_{\mu\nu}\left[m^2 - P^2 + q^2/4\right]\Big] =$$

$$= 4P_\mu P_\nu - q_\mu q_\nu + 2\eta_{\mu\nu}\left(m^2 - P^2 + q^2/4\right)$$

Finally, by combining the denominators using the standard trick:

$$\frac{1}{AB} = \int_0^1 d\alpha \, \frac{1}{\left[\alpha A + (1-\alpha) B\right]^2}$$

and replacing, after performing the shift of the integration variable:

$$\int P_\mu P_\nu \longrightarrow \eta_{\mu\nu} \int \frac{1}{2} P^2$$

you will show as an exercise that one obtains:

$$\Pi^{W}_{\mu\nu}(m) = \int_0^1 d\alpha \int d^2 p \; \frac{2\eta_{\mu\nu}\left[\alpha(1-\alpha)q^2 + m^2\right] - 4\alpha(1-\alpha) q_\mu q_\nu}{\left[P^2 + \alpha(1-\alpha)q^2 - m^2\right]^2}$$

what is extremely important here is that the integral has become finite! This you could have seen already from the expression of the numerator, where the only P^2 term, that are the ones leading to a divergence, are

$$2\left[2 P_\mu P_\nu - P^2\right]$$

which vanishes by the replacement $PP \to \eta P^2/2$.

Of course the replacement must be performed after the shift, but this does not affect the cancellation of the divergence.

So the result, for $\Pi(m)$ and $\Pi(M_{PV})$ separately, is finite, we do not need to subtract (as it happens normally) the divergences in the two terms in order to get a finite result. It might seem, then, that we did not need a regulator at all! This is not the case, because for being allowed to perform the shift (and only after that the replacement) we first had to make the integral finite by the regulator. This of being superficially divergent but actually finite is a feature of all the anomalies, in any dimension.

One can eventually perform the integral in d^2p and obtain (exercise)

$$\Pi^{VV}_{\mu\nu}(m) = \frac{i\pi}{(2\pi)^2}\int_0^1 d\alpha \; \frac{2\eta_{\mu\nu}\left[\alpha(1-\alpha)q^2+m^2\right]-4\alpha(1-\alpha)q_\mu q_\nu}{m^2-\alpha(1-\alpha)q^2}$$

and eventually:

$$\Pi^{\text{reg}\,VV}_{\mu\nu} = \lim_{\substack{m\to 0\\ M\to\infty}}\left(\Pi^{VV}_{\mu\nu}(m)-\Pi^{VV}_{\mu\nu}(M)\right) = \frac{i}{\pi}\left(\frac{q_\mu q_\nu}{q^2}-\eta_{\mu\nu}\right)$$

Eventually, we get

$$\Pi_{\mu\nu} = \varepsilon_{\mu\rho}\, \Pi^{W,\,\mathrm{reg}\,\rho}{}_{\nu} = \frac{e}{\pi}\left(\frac{\tilde{q}_\mu q_\nu}{q^2} - \varepsilon_{\mu\nu}\right)$$

where $\tilde{q}_\mu \equiv \varepsilon_{\mu\nu}\, q^\nu$.

As expected, given that the regulator preserved the vector symmetry, the result respects the vector Ward Identity:

$$q^\nu\, \Pi_{\mu\nu} = 0$$

but <u>does not respect</u> the axial one:

$$\underbrace{q^\mu\, \Pi_{\mu\nu} = \frac{e}{\pi}\,\tilde{q}_\nu \neq 0}_{\displaystyle \downarrow \atop \text{Anomaly}}$$

From A. SCHWIMMER'S LECTURES

• More on the Chiral Anomaly in 2d •

We have found a violation of the Axial Ward identity, this was not surprising because we have been working with a regulator that preserved the Vector symmetry while breaking the axial. We would have found the opposite with a regulator preserving the axial and breaking the vector. Remember, out of technicalities like the Pauli-Villard method, or dimensional regularization, that a regulator is nothing but a recipe to systematically perform the subtractions that are needed to make the integrals finite. Suppose you always regulate the integrals with the cutoff Λ, this you can always do even if the integral, as in the Pauli-Villard method, was already finite. Performing the integral of the two pieces separately, for $\Lambda \to \infty$, in our calculation we have:

$$\Pi^{VV}_{\mu\nu} = \Pi^{VV}_{\mu\nu}(m) - \Pi^{VV}_{\mu\nu}(M_{PV}) = C\log(\Lambda)\,\eta_{\mu\nu} + $$

$$+ \text{"finite}(m) - C\log(\Lambda)\eta_{\mu\nu} - \text{"finite}(M_{PV})\text{"}$$

the regulator correspond to a specific subtraction

$$\text{"finite}(M_{PV})\text{"} = \gamma\,\eta_{\mu\nu} + \frac{1}{M_{PV}^2} q_\mu q_\nu + \cdots$$

of the local terms. In this case, the only relevant local term is $\eta_{\mu\nu}$. The ones with more qs like $q_\mu q_\nu$ must have a $1/\Lambda$ (or $1/M_{PV}$) by dimensionality, this is why we cannot subtract them. By a different choice of the subtraction (i.e., of the regulator) we can then change our result into:

$$\Pi_{\mu\nu} = \varepsilon_{\mu\rho} \Pi^{W\rho}{}_{\nu} = \frac{e}{\pi}\left(\frac{\tilde{q}_\mu q_\nu}{q^2} - a\,\varepsilon_{\mu\nu}\right)$$

Imposing conservation of the Vector:

$$(1-a)\,\tilde{q}_\mu = 0 \quad \Rightarrow a \neq 1$$

Imposing conservation of the Axial:

$$+a\,\tilde{q}_\nu = 0 \quad \Rightarrow \quad a = 0$$

You see that <u>both</u> <u>conditions</u> <u>cannot</u> <u>be</u> <u>restored</u>

Therefore "The Anomaly can be moved from one place to the other, but not canceled, by the choice of the regulator, i.e. by adding a local finite counterterm to the correlator."

③

To appreciate how robust this point is, forget about our explicit calculation, and think more abstractly to what we might have obtained as a result. Imposing the various symmetries of our problem, $\Pi_{\mu\nu}(q)$ is very constrained:

1) Lorentz Invariance
2) Parity

$$\left.\right\} \Rightarrow \Pi_{\mu\nu} = E(q^2)\tilde{q}_\mu q_\nu + C(q^2)\,\varepsilon_{\mu\nu}$$

3) Vector Conservation $\Rightarrow E(q^2)\cdot q^2 + C(q^2) = 0$

4) Axial conservation $\Rightarrow C(q^2) = 0$

This shows that there is a clash among the various symmetries, that simply cannot be imposed simultaneously. This is the (regulator-independent) meaning of the Anomaly.