Sheet 9

Due: 24/05/11

Question 1 [*The* $SU(2)_L \times SU(2)_R$ σ *model*]:

Consider a theory for which the strong isospin (T) is a global $SU(2)$ symmetry.This theory contains the following fields: an isospin triplet of pions $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, with $T = +1$ and $T_3 = -1, 0, +1$, an isoscalar field σ with $T = 0$ and an isodoublet of nucleons $N = (p, n)$ for which $T = \frac{1}{2}$ and $T_3 = \pm\frac{1}{2}$.
For this theory we consider the following Lagrangian

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi^a\partial^\mu\pi^a\right) + \bar{N}i\gamma^\mu\partial_\mu N + g\bar{N}(\sigma + i\tau^a\pi^a\gamma_5)N \\ & + \frac{\mu^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2, \end{aligned} \tag{1}$$

with $\tau^a (a = 1, ..3)$, the Pauli matrices.

(i) Show that the Lagrangian is invariant under the following infinitesimal $SU(2)$ global transformation, called $SU(2)_V$ (V: vector)

$$\begin{aligned} \sigma &\rightarrow \sigma' = \sigma \\ \pi^a &\rightarrow \pi'^a = \pi^a + \varepsilon^{abc}\alpha^b\pi^c \\ N &\rightarrow N' = N - i\alpha^a\frac{\tau^a}{2}N, \end{aligned} \tag{2}$$

where α_i are infinitesimal space-independent group parameters associated to $SU(2)_V$. Find the Noether currents J^a_μ associated with this symmetry.

(ii) Show that the Lagrangian is also invariant under the infinitesimal axial $SU(2)$ transformation, called $SU(2)_A$

$$\begin{aligned} \sigma &\rightarrow \sigma' = \sigma + \beta^a\pi^a \\ \pi^a &\rightarrow \pi'^a = \pi^a - \beta^a\sigma \\ N &\rightarrow N' = N + i\beta^a\frac{\tau^a}{2}\gamma_5 N, \end{aligned} \tag{3}$$

where β_i are infinitesimal space-independent group parameters associated to $SU(2)_A$. Find the Noether currents A^a_μ associated with this other symmetry.

(iii) Verify that the corresponding charges

$$Q^a = \int \mathrm{d}^3x J^a_0(x) \qquad \text{and} \qquad Q^{5a} = \int \mathrm{d}^3x A^a_0(x) \tag{4}$$

are conserved, i.e. show that $\mathrm{d}Q^a/\mathrm{d}t = 0$ and $\mathrm{d}Q^{5a}/\mathrm{d}t = 0$, and prove that they generate the $SU(2)_L \times SU(2)_R$ algebra,

$$[Q^a, Q^b] = i\varepsilon^{abc}Q^c \tag{5}$$
$$[Q^a, Q^{5b}] = i\varepsilon^{abc}Q^{5c} \tag{6}$$
$$[Q^{5a}, Q^{5b}] = i\varepsilon^{abc}Q^c. \tag{7}$$

(iv) For $\mu^2 > 0$ spontaneous symmetry breakdown will happen, since the minimum of the potential is at

$$\sigma^2 + \boldsymbol{\pi}^2 = v^2 \qquad \text{with} \qquad v = (\mu^2/\lambda)^{\frac{1}{2}}. \tag{8}$$

Show that, if we choose

$$\langle 0|\pi^a|0\rangle = 0 \qquad \text{and} \qquad \langle 0|\sigma|0\rangle = v, \tag{9}$$

and write the Lagrangian in terms of the shifted field $\sigma' = \sigma - v$, the $\boldsymbol{\pi}$s remain massless while the nucleons and the isoscalar acquire a masses $m_N = gv$ and $m_\sigma = \sqrt{2}\mu$, respectively.

(v) Show that

$$[Q^{5a}, \pi^b] = -i\sigma\delta^{ab}, \tag{10}$$

and that the choice of eq.(9) implies that the axial charges Q^{5a} do not annihilate the vacuum while the Q^as do. This means that the axial $SU(2)$ symmetry ($SU(2)_A$) is broken and therefore that the $SU(2)_L \times SU(2)_R$ symmetry is broken spontaneously into the $SU(2)_V$ symmetry generated by the charges Q^a.

Hint: Show that $\langle 0|A^a_\mu(0)|\pi^a\rangle \neq 0$ and $Q^a|0\rangle = 0$.