Sheet 7

Due: 10/05/11

Question 1 [*Scalar QCD*]: Consider the theory of a complex scalar field interacting with gauge bosons of a group G, described by the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \ , \tag{1}$$

where ϕ lives in a given representation R of G with generators T^a_R and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

$$D^\mu = \partial^\mu - i g T^a_R A^\mu_a \ . \tag{3}$$

The goal of this exercise is to derive the Feynman rules in the gauge $\partial_\mu A^\mu_a = 0$.

(i) Add the R_ξ gauge-fixing Lagrangian $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} \partial^\mu A^a_\mu \partial^\nu A^a_\mu$ and the Faddeev-Popov ghost Lagrangian $\mathcal{L}_{\text{FP}} = -\bar{c}^a \partial_\mu D^\mu_{ab} c^b$, where $D^{ab}_\mu = \delta^{ab}\partial_\mu - i g A^c_\mu (T^c_A)^{ab}$, with T_A being the generators of G in the adjoint representation. After adding the source terms for gauge bosons and scalar fields, one can write the generating functional as

$$Z[J_A, J_\phi, J_{\phi^\dagger}] \propto \int \mathcal{D}A\, \mathcal{D}\phi\, \mathcal{D}\phi^\dagger\, \mathcal{D}\bar{c}\, \mathcal{D}c \exp(i S_{\text{free}} + i S_{\text{int}} + i S_{\text{src}}) \ , \tag{4}$$

where S_{free} contains the kinetic terms, S_{int} contains the interaction terms, while external sources are contained in S_{src}. Determine S_{free}, S_{int} and S_{src} explicitly.

(ii) Rewrite S_{free} and S_{int} in momentum space.

(iii) Derive the field propagators from S_{free}.

(iv) Derive the interaction vertices from S_{int}.

(v) Compute the contribution of this scalar field to the β-function, and show that the full β function for this theory is

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C_2(G) - \frac{1}{3} C(R) \right) \ , \tag{5}$$

where $C_2(G)$ and $C(R)$ are the Casimir operators in the adjoint and R representations, respectively.

Question 2 [*Asymptotic symmetry*]: Consider the Lagrangian with two scalar fields ϕ_1 and ϕ_2

$$\mathcal{L} = \frac{1}{2}\Big((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\Big) - \frac{\lambda}{4!}\big(\phi_1^4 + \phi_2^4\big) - \frac{2\rho}{4!}\phi_1^2\phi_2^2 \ . \tag{6}$$

Notice that, for the special value $\lambda = \rho$, this Lagrangian has a manifest $O(2)$ invariance, rotating the two fields into one another.

(i) Working in four dimensions, find the β function for the two coupling constants λ and ρ, to leading order in the coupling constants.

(ii) Write the renormalisation group equation for the ratio of couplings ρ/λ. Show that, if $\rho/\lambda < 3$ at a renormalisation scale M, this ratio flows towards the value $\rho = \lambda$ at large distances. Thus the $O(2)$ internal symmetry appears asymptotically.

(iii) Write the β functions for λ and ρ in $4-\epsilon$ dimensions. Show that there are non-trivial fixed points of the renormalisation group flow at $\rho/\lambda = 0, 1, 3$. Which is the most stable? Sketch the pattern of coupling constant flows.

Question 3 [*The Gross-Neveu Model*]: The Gross-Neveu model is a two spacetime dimensional model of fermions with a discrete chiral symmetry. Its Lagrangian is given by

$$\mathcal{L} = i\,\bar{\psi}_i\,\partial\!\!\!/\,\psi_i + \frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2 \ , \tag{7}$$

where $i = 1, \ldots, N$ labels the different fermions. The kinetic term of the two-dimensional fermions is built from matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be taken to be the 2×2 matrices

$$\gamma^0 = \sigma^2 \ , \qquad \gamma^1 = i\,\sigma^1 \ , \tag{8}$$

where σ^i are the Pauli matrices. Define

$$\gamma^5 = \gamma^0\gamma^1 = \sigma^3 \ . \tag{9}$$

(i) Show that the theory is invariant with respect to

$$\psi_i \mapsto \gamma^5\psi_i \ , \tag{10}$$

and that this symmetry forbids the appearance of a fermion mass term.

(ii) Compute $\beta(g)$ and show that the model is asymptotically free.