Solid State Theory
Exercise 11

Transport in metals

Exercise 11.1 The f-sum rule

The f-sum rule is a consequence of the particle conservation in a given system

$$\int_0^\infty d\omega' \sigma_1(\omega') = \frac{\pi e^2 n}{2m}. \tag{1}$$

It is a useful tool to check the consistency of any approximative treatment. In the following we want to derive the f-sum rule for an electronic system with the (one-particle) Hamilton operator

$$\mathcal{H} = \mathcal{H}_0 + U(\mathbf{r},t), \quad \mathcal{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r}), \quad U(\mathbf{r},t) = A_0 e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + A_0^* e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}. \tag{2}$$

The time-dependent terms induce transitions between the eigenfunctions of $\mathcal{H}_0$.

a) Use Fermi's golden rule to compute the transition rate $W(\mathbf{q},\omega)$ of the many-particle system.

The transitions between the states of $\mathcal{H}_0$ lead to an energy dissipation of the amount $\hbar\omega W(\mathbf{q},\omega)$. The real part of the conductivity, $\sigma_1(\mathbf{q},\omega)$, is related to the energy dissipation by

$$\sigma_1(\mathbf{q},\omega) = \frac{1}{2}\frac{e^2}{q^2}\frac{\hbar\omega W(\mathbf{q},\omega)}{|A_0|^2 V}. \tag{3}$$

b) Compute the double commutator

$$\left[e^{-i\mathbf{q}\cdot\mathbf{r}}, [\mathcal{H}_0, e^{i\mathbf{q}\cdot\mathbf{r}}]\right] \tag{4}$$

and derive the following equation

$$\frac{1}{V}\sum_{\alpha\beta} f(E_\alpha)\left|\langle\psi_\beta|e^{i\mathbf{q}\cdot\mathbf{r}}|\psi_\alpha\rangle\right|^2 (E_\beta - E_\alpha) = \frac{\hbar^2 q^2}{2m}\frac{n}{2} \tag{5}$$

where E_α and ψ_α are the eigenvalues and eigenfunctions of $\mathcal{H}_0$.

c) Combine Eq. (3) and Eq. (5) in order to proof the f-sum rule Eq. (1).

Exercise 11.2 Penetration depth in a superconductor

We consider a superconductor with a normal-conducting component ρ_n and a superconducting component ρ_s, where $\rho = \rho_n + \rho_s$ is the total electron density. The conductivity of the system is given by the conductivity of the two components, $\sigma = \sigma_n + \sigma_s$, with

$$\sigma_n(\omega) = \rho_n \frac{e^2}{m} \frac{\tau}{1 - i\omega\tau} \quad \text{and} \quad \sigma_s(\omega) = i\rho_s \frac{e^2}{m(\omega + i0^+)}. \tag{6}$$

The density of the superconducting component depends on the temperature in the following way (Gorter-Casimir two-fluid model)

$$\rho_s(T) = \rho \left[1 - \left(\frac{T}{T_c} \right)^4 \right]. \tag{7}$$

a) Use the expression for the dielectric constant of a metal $\varepsilon(\omega) = 1 + (4\pi i/\omega)\sigma(\omega)$ in order to compute the penetration depth $\delta(\omega, T)$ in the limit $\omega\tau \ll 1$.

b) Plot the penetration depth $\delta(\omega, T)$ for small ω as a function of the frequency ω and temperature T. Discuss the limits $T \to T_c$ and $T \to 0$.

c) Show that the f-sum rule also holds for a superconductor with conductivity given in Eq. (6).

Exercise 11.3 Reflectivity of Simple Metals and Semiconductors

Use the expression for the Drude conductivity,

$$\sigma(\omega) = \frac{\omega_p^2}{4\pi} \frac{\tau}{1 - i\omega\tau}, \tag{8}$$

to obtain an expression for the reflectivity $R(\omega)$ of a simple metal or semiconductor using the connection between $\sigma(\omega)$ and $R(\omega)$ given in section 5.2.2. of the lecture notes. To take into account the effect of the bound (or core) electrons, use as a phenomenological ansatz for the dielectric function

$$\epsilon(\omega) = \epsilon_\infty + \epsilon_{\text{Drude}}(\omega) - 1. \tag{9}$$

Here ϵ_∞ is assumed to be constant in the frequency range of interest, related to the fact that the energy scale for exciting core electrons is much higher than the typical energy scales for the itinerant electrons. Plot the reflectivity for the cases $\epsilon_\infty = 1$ and $\epsilon_\infty = 20$ and $\tau\omega_p = \infty, 40, 2$!

Usually, ϵ_∞ is much larger in semiconductors than in metals. Can you think about possible explanations for this behavior?