Quantum Field Theory II
Exercise sheet 2

FS 10
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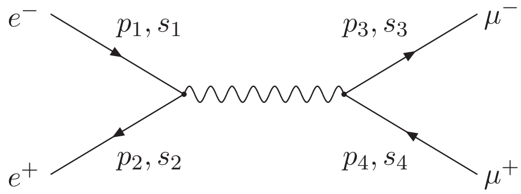
Exercise 2.1 e^+e^- annihilation

In this exercise, we calculate the leading order unpolarised cross section for the QED-process $e^+e^- \to \mu^+\mu^-$. For simplicity, all particles will be treated to be massless. This cross section is the denominator of the famous R *coefficient* whose value determines the number of quarks with a mass smaller than the collision energy. At tree-level, the process is given by a single Feynman Diagram:



a) Use the QED Feynman rules (QFT I, p. 191) to calculate the matrix element M for this diagram. External lines are to be truncated.

b) Square the matrix element. Here, you will have to make use of the *polarisation sums* for massless fermions:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} = \sum_s v^s(p)\bar{v}^s(p)$$

Furthermore, it is convenient to use the *Mandelstam variables*:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

The result is:

$$|M|^2 = \frac{2e^4(t^2 + u^2)}{s^2}$$

Hint: You need to *sum* over final state spins and *average* over initial state spins.

c) Finally, compute the *total cross section* σ as seen in today's lecture (i.e. integrating over the 2-particle phase-space):

$$d\sigma = \frac{1}{2E_1 2E_2 |\vec{v}_1 - \vec{v}_2|} \frac{d^3p_3}{2E_3(2\pi)^3} \frac{d^3p_4}{2E_4(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, |M|^2$$

Your result should be

$$\sigma = \frac{4\pi\alpha^2}{3s}, \quad \text{where} \quad \alpha = \frac{e^2}{4\pi}.$$

Hint: Work in the *center-of-mass frame* of the two incoming particles and exploit the symmetry of the process.

Exercise 2.2 Dual field strength tensors

a) In an abelian gauge theory, consider the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Show that

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K^\mu,$$

with $K^\mu = \epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$.

Hint: Use that contractions of the ϵ tensor with a symmetric tensor vanish.

b) In a non-abelian gauge theory, consider the dual tensor $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$. Show that

$$Tr\left(G^{\mu\nu}\tilde{G}_{\mu\nu}\right) = \partial_\mu K^\mu,$$

with $K^\mu = \epsilon^{\mu\nu\rho\sigma}Tr\left[G_{\nu\rho}A_\sigma + \frac{2}{3}igA_\nu A_\rho A_\sigma\right]$.

Hint: Use the cyclicity of the trace.

Remark: The dual field tensor in QED corresponds to the interchange of E and B; one often says that E and B are dual.
The term $F^{\mu\nu}\tilde{F}_{\mu\nu}$ is called the *θ-term*, $Tr(G^{\mu\nu}\tilde{G}_{\mu\nu})$ goes by the name of *$\bar{\theta}$-term*. They are manifestly Lorentz invariant objects. In QED, it corresponds to $E \cdot B$. These terms are in fact CP-violating and therefore, their coupling constant must be very small (due to experimental constraints on CP-violation). The apparent lack of a reason for the $\bar{\theta}$-term to be so small is called the *strong CP problem*.
The θ-term also arises in the context of anomalies where they correspond to the right hand side of the anomaly equation (see e.g. Peskin & Schröder, (19.45)). Anomalies are symmetry violations that only arise at the one-loop level. They are very important in quantum field theory. For instance, the masses of the nucleons are due to an anomaly in the energy-momentum tensor.