Cross sections

Predictions for experiments: cross sections and decay rates

From QFT 1, p. 144

$$S_{ji} = \langle \text{out } \{k_j\} | \{p_i\} \text{ in} \rangle \qquad (11.48) \quad (1)$$

is the probability amplitude for a transition from $|\{p_i\}\rangle$ to $|\{k_j\}\rangle$

S-matrix - or scattering matrix element.*

The scattering probability is

$$P_{ji} = P(\{p_i\} \to \{k_j\}) = |S_{ji}|^2 \qquad (2)$$

The S-matrix is unitary

$$(SS^+)_{ij} = S_{ik} S^+_{kj} = \delta_{ij} \qquad (3)$$

One writes often

$$S = 1 + iT \qquad (4)$$

where T is called the transition matrix and includes non-diagonal processes. Note that

$$1 = SS^+ = (1+iT)(1-iT^+) = 1 + i(T - T^+) + TT^+$$

$$2\,\mathrm{Im}\, T = TT^+ \qquad (5)$$

(important phenomenologically !!)

The P_{ij} are of interest experimentally, however not directly. What is measured are cross-sections and other rates.

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* S_{ji} are the matrix elements of an operator $\hat{S}$ that acts on states. One sets $\langle \{k_j\} | \hat{S} | \{p_i\} \rangle = \langle \text{out } \{k_j\} | \{p_i\} \text{ in} \rangle = S_{ji}$

Relating the S-matrix to calculable quantities

In QFT I, p. 150, it was shown that

$$S_{ji} = \langle \text{out } \{k_j\} | \{p_i\} \text{ in} \rangle \sim \tag{6}$$

$$= \prod (\ldots\ldots) \, \tilde{G}(\{k_j\}, \{p_i\}) \qquad (11.76)$$

which shows that in order to get S, we need $\tilde{G}$ (Greens functions). From the considerations in 12.6 (QFT I, p. 167) we find that

$$\tilde{G}(\{k_j\}, \{p_i\}) \sim (2\pi)^4 \, \delta^4(\Sigma k_j - \Sigma p_i) \, F_{ji} \tag{7}$$

(overall momentum conservation).

Therefore it is reasonable to equally write $S \sim (2\pi)^4 \ldots$ and one sets

$$S_{ji} = (2\pi)^4 \, \delta^4(\Sigma k_j - \Sigma p_i) \, i \, \mathcal{M}_{ji} \tag{9}$$

$\mathcal{M}$ is called invariant matrix element and is calculated from F_{ji} in (7) using Feynman rules.

Example: From QFT I, p. 169 one gets

$$G_d = \frac{(-i\lambda)\,(i)^4\,(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)}{\prod_i (p_i^2 - m^2 + i\delta)} \qquad \begin{matrix}(10)\\(12.79)\end{matrix}$$

From (11.76) one has for the case d

$$S_{ji} = \prod_i \left[\frac{(p_i^2 - m^2)}{i\sqrt{\tilde{Z}}} \right] G_d \qquad \begin{matrix}(11.76)\\(11)\end{matrix}$$

This tells us that

$$S_{ji} = (-i\lambda)\,(2\pi)^4\,\delta^4(\quad) \tag{12}$$

Comparison of (9) and (12):

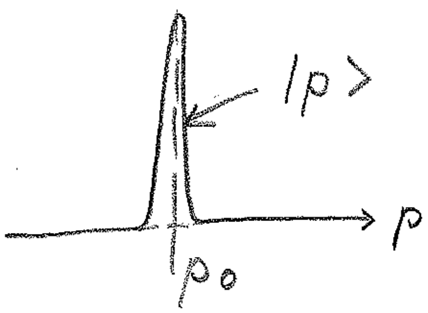
$$i\mathcal{M}_{ji} = -i\lambda \qquad \mathcal{M} = -\lambda \tag{13}$$

In this way one can calculate any invariant matrix element. Examples follow.

Relating P_{ji} to Observables

a) A mathematical problem

When calculating $P \sim |S|^2$ and using (9), one finds $(\delta^4)^2$ which is undefined mathematically (one power of δ^4 is ok is we can integrate over momenta). The problem stems from the fact that we consider only sharp momenta states $|p_0\rangle \sim \delta(p-p_0)$. When looking at the norm, $\int dp \, ||p_0\rangle|^2 = 1$, we recognize that the value at p_0 must be infinite, and this leads to the problem.



As usual, the experimental set-up resolves the problem. In reality, particles have a momentum spread, that is, they come in so called wave packets.

We write therefore for a state

$$|\varphi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k) |k\rangle \tag{14}$$

To understand the factors, remember (QFT I, p. 45)

$$|k\rangle = a_k^+ |k\rangle \qquad \text{and} \qquad [a_{k_1}, a_{k_2}^+] = (2\pi)^3 (2E_{k_1}) \delta^3(\vec{k}_1 - \vec{k}_2)$$

This yields

$$\langle k_1 | k_2 \rangle = \langle 0 | a_{k_1} a^\dagger_{k_2} | 0 \rangle = \langle 0 | a_{k_1} a^\dagger_{k_2} - a^\dagger_{k_2} a_{k_1} + a^\dagger_{k_2} a_{k_1} | 0 \rangle$$

$$= \langle 0 | (2\pi)^3 2E_{k_1} \delta(k_1 - k_2) | 0 \rangle = (2\pi)^3 2E_{k_1} \delta(k_1 - k_2)$$

From this

$$\langle \phi | \phi \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \langle k | q \rangle \frac{\phi^*(k)\phi(q)}{\sqrt{2E_k 2E_q}}$$

$$= \int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 \qquad (15)$$

Thus if $\langle \phi | \phi \rangle = 1$, then the "sum" (integral) of all probabilities $|\phi(k)|^2$ is 1, as desired.

Using (+) for all states, we can set

$$\langle \text{out } \varphi_j | \varphi_i \text{ in} \rangle = \qquad (16)$$

$$\int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \cdots \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{\phi(p_1)}{\sqrt{2E_{p_1}}} \cdots \frac{\phi(k_n)}{\sqrt{2E_{k_n}}} \langle \text{out} \{k_j\} | \{p_i\} \text{in} \rangle$$

We can simplify by taking the final state to consist of n particles with momenta in small regions $d^3k_1, d^3k_2 \ldots$ Then the probability to go from the state $|\varphi_i \text{ in}\rangle$ to the final state is

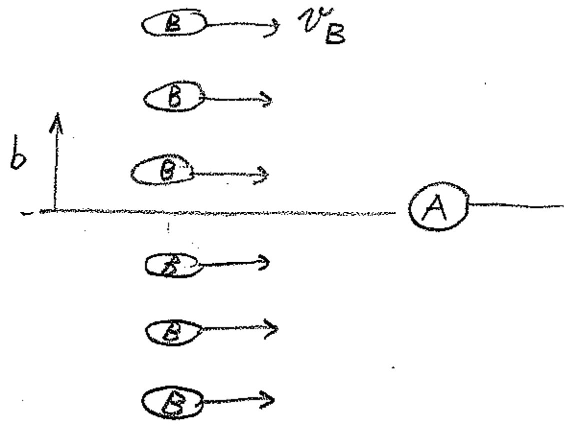
$$dP_{ji} = \prod_{a=1}^{n} \left(\frac{d^3k_a}{(2\pi)^3} \frac{1}{2E_a} \right) |\langle \text{out } \{k_j\} | \varphi_i \text{ in} \rangle|^2 \qquad (17)$$

where $|\varphi_i \text{ in}\rangle$ consists of wave packets as in (+).

$$|\varphi_i\rangle \longrightarrow \begin{cases} \{k_1, k_1 + dk_1\} \\ \{k_2, k_2 + dk_2\} \\ \vdots \\ \{k_3, k_3 + dk_3\} \end{cases}$$

In the physical situation we may have a target particle A on which many incident particles B impinge. The initial state is (see comments)

$$\sim \int \frac{dp_A \, dp_B}{(2\pi)^6 \sqrt{2E_A E_B}} \phi_A \phi_B e^{-ibp_B} |\{p_i\} \text{in}\rangle \qquad (18)$$



The number N of scatterings is

$$N = \int d^2b \, n_B \, P(b) \qquad (19)$$

where n_B is the number of B particles per unit area

The cross section σ is defined by

$$N = \rho_B \, l_B \, \rho_A \, l_A \cdot F \cdot \sigma \qquad (20)$$

where F is the common area to the B bunches and the target, ρ_B and l_B the density and length of the B bunches and likewise for A.

If the densities are constant over the area F, then $N_A = \rho_A l_A F$, etc. and

$$N = N_A \cdot N_B \frac{\sigma}{F} \qquad (21)$$

Comparison of (xx) and (xxx) gives

$$\sigma = \frac{N \cdot F}{N_A N_B} = \frac{N}{n_B} \frac{1}{N_A} = \frac{\int d^2b \, n_B P(b)}{n_B \cdot N_A} \qquad (22)$$

n_B

And if $N_A = 1$

$$\sigma = \int d^2b \, n_B \, P(b)$$

From this one can get

$$d\sigma = \prod_a \left(\frac{d^3k_a}{(2\pi)^3 2E_a} \right) \int d^2b \prod_i \left(\int \frac{dp_i \, \phi(p_i)}{(2\pi)^3 \sqrt{E_i 2}} \int \frac{d\bar{p}_i \, \bar{\Phi}(\bar{p}_i)}{(2\pi)^3 \sqrt{E_i 2}} \right) \qquad (23)$$
$$e^{ib(\bar{p}_B - p_B)} \langle \text{out} \{k_j\} | \{p_i\} \text{ in} \rangle \langle \text{out} \{k_j\} | \{\bar{p}_i\} \text{ in} \rangle^*$$

We now rewrite the matrix elements $\langle \quad \rangle$ using (9)

as

$$\langle \qquad \rangle = i\mathcal{M}(2\pi)^4\,\delta^4\left(\Sigma k_j - \Sigma p_i\right) \tag{24}$$

$$\langle \qquad \rangle^* = -i\mathcal{M}(2\pi)^4\,\delta^4\left(\Sigma k_j - \Sigma \bar{p}_i\right) \tag{25}$$

Collecting terms in various directions:

transverse

$$\int d^2b\, e^{ib(\bar{p}_B - p_B)} = (2\pi)^2\,\delta^2\left(\bar{p}_B^{\perp} - p_B^{\perp}\right) \tag{26}$$

$$\int \frac{d\bar{p}_A\, d\bar{p}_B}{(2\pi)^6\; 2\bar{E}_A\, 2\bar{E}_B}\,\delta\left(\bar{p}_A + \bar{p}_B - \Sigma k_j^z\right)\delta\left(\bar{E}_A + \bar{E}_B - \Sigma E_j\right) \tag{27}$$

($\bar{p}_A, \bar{p}_B$ go along z)

$$= \int d\bar{p}_A\, \delta\left(\sqrt{\bar{p}_A^2 + m_A^2} + \sqrt{\bar{p}_B^2 + m_B^2} - \Sigma E_j\right)\Big|_{k_B = \Sigma k_j - p_A}$$

$$= \frac{1}{\left|\dfrac{\bar{p}_A}{\bar{E}_A} - \dfrac{\bar{p}_B}{E_B}\right|} = \frac{1}{|v_A - v_B|} \tag{28}$$

where $|v_A - v_B|$ is the relative velocity in the Lab frame

One can re-express this factor (see below)

We return to (23). Since the ϕ are centered around p_A, p_B, we can essentially take out all smooth functions of these arguments. Then

$$d\sigma = \Pi\left(\frac{d^3k_a}{(2\pi)^3 2E_a}\right)\frac{|\mathcal{M}|^2}{2E_a 2E_B |v_A - v_B|}\int \frac{d^3p_A}{(2\pi)^3}\frac{d^3p_B}{(2\pi)^3}\,|\phi(p_A)|^2|\phi(p_B)|^2$$
$$(2\pi)^4\,\delta\left(p_A + p_B - \Sigma k_j\right) \tag{29}$$

$$\int \delta\left(\sqrt{\bar{p}_A^2 + m_A^2} - \ldots\right) = \frac{1}{\dfrac{\bar{p}_A}{\bar{E}_A}\ldots}$$

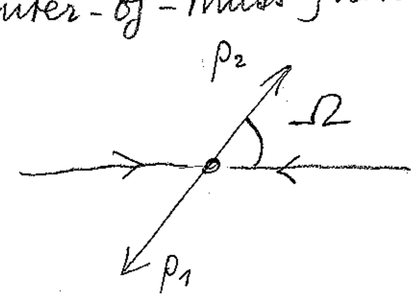
In the next step, we take $p_A + p_B$ essentially fixed to the experimental value (this is justified, if the final momenta are measured with less precision than the variation of $p_A + p_B$). Then the δ-function goes out of the integral and with (15) we have

$$d\sigma = \frac{\prod \left(\frac{d^3 k_j}{(2\pi)^3} \frac{1}{2E_j} \right)}{2E_A \, 2E_B \, |v_A - v_B|} \, |\mathcal{M}|^2 \, (2\pi)^4 \, \delta^4 \left(p_A + p_B - \sum p_f \right) \tag{30}$$

This is the final expression. It has the necessary invariance properties (under Lorentz-transformation). For one $\left(\frac{d^3 k}{2E} \right)$ is invariant. But also $E_A E_B |v_A - v_B|$ is invariant under boost in z-direction and can be written (with proper use!) as

$$E_A E_B |v_A - v_B| = \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2} \tag{31}$$

For two particles in the final state, one can go to the center-of-mass frame



$$\frac{d\sigma}{d\Omega} = \frac{1}{2 \cdot 2 E_A E_B |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 \, 4 E_{CM}} |\mathcal{M}|^2 \tag{32}$$

A similar formula holds for the decays of a single particle. One finds

$$d\Gamma = \frac{1}{2 m_A} \left(\prod \frac{d^3 k_i}{(2\pi)^3} \frac{1}{2E_i} \right) |\mathcal{M}(A \to \{k_j\})|^2 (2\pi)^4 \delta^4 \left(p_A - \sum k_i \right)$$

(see Peskin/Schröder, p. 107 etc)

Comments

The formula (x x x) for N can be understood as follows $(\rho_B l_B)$, $(\rho_A l_A)$ are the numbers of particles per unit area. In the area F there are $\rho_B l_B F$ particles B (or $\rho_A l_A F$ particles A). The naive product $(\rho_A l_A F)(\rho_B l_B F)$ for N is wrong because this would mean that all particles scatter. We can correct by adding a factor ε for the probability. Then we can call (εF) the cross section, $\sigma = (\varepsilon F)$. The reason we calculate (measure) σ rather than ε is that in experiments only the face F of the bunch counts, and not its length ( )

The factor e^{-ibp_B} in (18) comes because the wave packet B is centered at a distance b.
To see this: The wave packets are at $x = vt e_x$ &
$\varphi(x) \sim \int dp\, e^{-ipx} f(p)$. If $x = vte_x + b$, the extra factor e^{-ipb} must be added.

Eq. (23). The 4-velocity v_μ is $(\gamma, \gamma\vec{v})$ $(c=1)$. Then $p_\mu = m v_\mu$, $p_0 = E = m\gamma$. Thus $\vec{P}/E = \vec{v}$.