BRST - summary

We have seen that the "total" $\mathcal{L}$-density

$$\mathcal{L}_{cl} + \bar{\eta}^a \Delta^a + \omega^a G^a + \frac{\xi}{2} G^a G^a$$

(eq. 276, p. 64) has the form (eq. 289, p. 68)

$$\mathcal{L}_{cl} + \delta_\theta \left[\left(-\frac{g}{\theta} \right) \bar{\eta}^a \left(G^a + \frac{\xi}{2} \omega^a \right) \right]$$

$$\equiv \mathcal{L}_{cl} + s g K \qquad (B-1)$$

(see also eq. above 284). Thus $\mathcal{L}$ is an BRST-invariant plus a term in the image of the BRST transformation. So $\mathcal{L}$ is in the kernel modulo terms in the image. Since $s^2 = 0$, $\mathcal{L}$ is in cohomology of the BRST-trafo. Equally, the states of a physically sensible theory - they satisfy $Q|\alpha\rangle = 0$ correspond to the cohomology of the BRST-T. Since cohomology is a strong tool in mathematics, (B-1) makes strong statements on allowed Lagrangians and states. I suspect that all admissible states can be found in this way. The essential point is that the physical results are independent of K*, that is of the gauge. Since the ghosts are in K, and if there is a choice of K in which the ghost decouple, the ghosts decouple in general. For YM theories such a choice is $A_3 = 0$ (axial gauge).

(See p. 36 of Weinberg II and ref. 14 there)

---

* K could be a very general function of η, $\bar{\eta}$ and ω

External field methods – effective actions

One often considers situations with external classical fields, but where the "action" is quantum mechanical. Example: pair production by an external field. But it turns out that even without a real external field the methodology is useful, because it allows to take into accound all multiloop effects in an "effective" tree level action where vertices and propagators are taken from a quantum effective action. The effective tree level action is the one-particle irreducible (1PI) vacuum-to-vacuum transition amplitude in the external fields. This allows for a convenient way to show renormalizability.

We had

$$\begin{aligned} Z[J] &\cong \langle 0 \text{ out} \mid \text{in } 0 \rangle \\ &= \int \mathcal{D}\phi \, e^{i(S(\phi) + \int d^4x \, J(x)\phi(x))} \\ &= \exp(iW[J]) \end{aligned} \qquad (5.1)$$

where W generate the connected green's functions. In particular

$$\langle \phi \rangle_J(x) = \frac{\langle 0|\phi|0\rangle_J}{\langle 0|0\rangle_J} = \frac{\delta}{\delta J(x)} W[J] \qquad (5.2)$$

where $\langle \phi \rangle_J = 0$ if $J = 0$ to have only one-particle asymptotic states.
$\langle \phi \rangle_J$ is a function of J. On the other hand, we can

prescribe $\langle\phi\rangle_J$ and find the corresponding J. We define the quantum effective action

$$\Gamma[\langle\phi\rangle_J] = W[J] - \int d^4x \, J(x) \langle\phi\rangle_J \qquad (5.3)$$

which turns out to be the sum of all connected 1PI graphs in the presence of J.

We have

$$\frac{\delta\Gamma[\langle\phi\rangle_J]}{\delta\langle\phi\rangle_{J'}(y)} = -\int d^4x \, \frac{\delta J(x)}{\delta\langle\phi\rangle_{J'}(y)} \langle\phi(x)\rangle_J - J'(y)^{*} \qquad (5.4)$$

$$+ \int \frac{\delta W[J]}{\delta J(x)} \frac{\delta J(x)}{\delta\langle\phi\rangle_J'(y)} \, d^4x = - J'(y)$$

(In the second term there is the integral over x because we must "sum" over all $J(x)$).

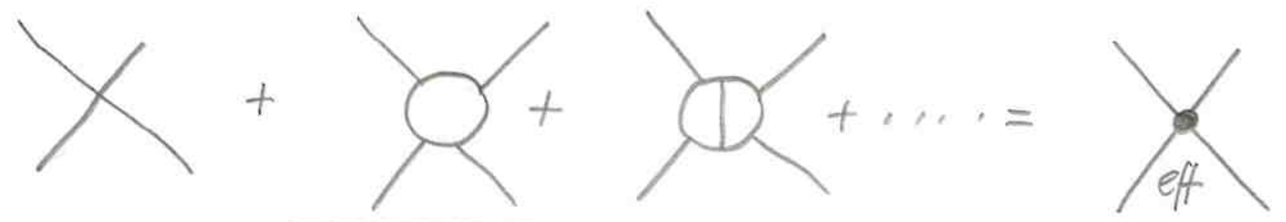
$$\langle\phi\rangle_J(x)$$

Thus

$$\frac{\delta\Gamma[\langle\phi\rangle_J]}{\delta\langle\phi\rangle_{J'}(y)} = -J'(y) = 0 \text{ if } J' = 0 \qquad (5.5)$$

The possible values for the "external" fields $\langle\phi\rangle_{J'}$, $J'=0$ are given an "Euler - Lagrange" type equation. Thus we call Γ the effective action, and $\langle\phi\rangle_{J'}$ the classical fields (when quantum corr. are included).

Example:



---

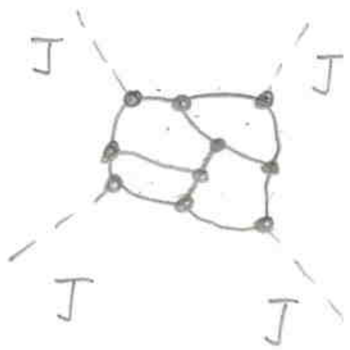
* $J'(x)$ is the value of J if $\langle\phi\rangle$ is fixed to $\langle\phi\rangle_{J'}$

Γ is to be considered a classical action and thus $W[J]$ is a sum of connected tree graphs with vertices and propagators for Γ. Loops are taken into account by using Γ instead of $S(\phi)$.

To see this, consider the inverse of (5.3) and replace Γ by $\frac{1}{g}\Gamma$ and exponentiate.

$$e^{iW_\Gamma[g,J]} = \int \partial\langle\phi\rangle_J \, e^{\frac{i}{g}\{\Gamma(\langle\phi\rangle) + \int d^4x \langle\phi\rangle_J^{(x)} J(x)\}} \tag{5.6}$$

In the usual expansion (5.1) the quadratic term in ϕ gives the (inverse of the) propagator (see Babis notes, eq. 180, p. 42. Thus a propagator gives a factor g, vertices from Γ a factor $\frac{1}{g}$.



V vertices ($V = 10$)
I lines ($I = 13$) (5.7)
L loops $= I - V + 1$ *

(Connected graphs)

The power in g is $g^{I-V} = g^{L-1}$. Then we set

$$W_\Gamma[J,g] = \sum_L g^{L-1} W_\Gamma^L[J] \tag{5.8}$$

where $W_\Gamma^L[J]$ is a L-loop part of $W[J,1]$.
The $W_\Gamma^L[J]$ have the symmetries of the full action separately (let this be!)

---

* L = # lines − # constraints (V) + 1 (overall mom. con)

Since g is arbitrary, we let $g \to 0$. Then $L = 0$ is dominant. We then have from (5.6)

$$\exp(iW_\Gamma[J,g]) \sim \exp\left\{\frac{i}{g}\left(\Gamma[\langle\phi\rangle_J] + \int d^4x \langle\phi\rangle_J(x) J(x)\right)\right. \tag{5.9}$$

because $\langle\phi\rangle_J$ is a stationary point of the exponent.
(Recall, $\int e^{F(\omega)} d\omega \sim e^{F(\omega_0) + \frac{\partial F}{\partial \omega}\delta\omega + \cdots} \sim e^{F(\omega_0)}$ if $\left.\frac{\partial F}{\partial \omega}\right|_{\omega=\omega_0} = 0$
We have $F \sim \Gamma(\langle\phi\rangle) + \langle\phi\rangle_J J(x)$, $\frac{\partial F}{\partial\langle\phi\rangle} = 0$ at $\langle\phi\rangle = \langle\phi\rangle_J$ because of (5.5)).

The leading power of g is g^{-1} *; in (5.8) it is for $L = 0$
Thus (take log of (5.9))

$$W_\Gamma^0[J] = \Gamma[\langle\phi\rangle_J] + \int d^4x \langle\phi\rangle_J(x) J(x) = W[J]$$

This means that

$$iW[J] = \int_{\substack{\text{connected}\\ \text{tree}}} \mathcal{D}\langle\phi\rangle \exp\left(i\Gamma[\langle\phi\rangle] + i\int \langle\phi\rangle_J J\, dx\right) \tag{5.10}$$

This says that $W[J]$ is the sum of tree graphs, but with $\Gamma[\langle\phi\rangle]$ instead of the usual action.
(5.10) means that $i\Gamma[\langle\phi\rangle]$ must be the sum of all 1PI connected graphs with arbitrary number of external lines corresponding to a factor $\langle\phi\rangle$. (and not momenta, for example).

From here, follow Babis, p. 75–80

---

* It is assumed, that all uncalculated terms are $g^0; g^1; ..$