Advanced Field Theory
Solution 13

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Exercise 13.1 The Partition Function in $\lambda\phi^4$ Theory

The exercise below consists of parts of chapters 1 and 3 of [Kap89].

1. Our starting point is the action

$$S = -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

which we rewrite by partial integration, the boundary terms vanish because ϕ is assumed to be periodic in τ and vanishing at spatial infinity, this gives us the form

$$S = -\frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^3x\, \phi\left[-\partial_\tau^2 - \Delta + m^2\right]\phi.$$

We expand the field in Fourier modes according to

$$\phi(\mathbf{x},\tau) = \left(\frac{\beta}{V}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty}\sum_{\mathbf{p}} e^{i(\mathbf{p}\mathbf{x}+\omega_n\tau)}\phi_n(\mathbf{p}).$$

The integration over $\mathbf{x}$ and τ results in $V\delta(\mathbf{p}+\mathbf{p}')$ and $\beta\delta(\omega_n+\omega_m)$ respectively. We remark that the reality condition $\phi = \phi^*$ means $\phi_{-n}(-\mathbf{p}) = \phi_n(\mathbf{p})^*$. We have rewritten the action in the form

$$S = -\frac{1}{2}\beta^2\sum_n\sum_{\mathbf{p}}\left[\omega_n^2 + \mathbf{p}^2 + m^2\right]|\phi_n(\mathbf{p})|^2.$$

We insert the action back into the partition function, ignoring prefactors of Z because these are irrelevant for thermodynamics:

$$Z \propto \int \mathcal{D}\phi \prod_n\prod_{\mathbf{p}} \exp\left[-\frac{1}{2}\beta^2\left(\omega_n^2+\mathbf{p}^2+m^2\right)|\phi_n(\mathbf{p})|^2\right].$$

We do the integration over field configurations, the phases of the $\phi_n(\mathbf{p})$ do give rise to an overall factor, for the integrations over the absolute values $|\phi_n(\mathbf{p})|$ we insert $\int\exp(-1/2\,a\,x^2)\mathrm{d}x \propto 1/\sqrt{a}$:

$$Z \propto \prod_n\prod_{\mathbf{p}}\left[\beta^2\left(\omega_n^2+\mathbf{p}^2+m^2\right)\right]^{-\frac{1}{2}}.$$

Of course, this is nothing else but the well-known functional determinant:

$$Z \propto \left(\det\left[-\partial_\tau^2 - \Delta + m^2\right]\right)^{-\frac{1}{2}}.$$

We evaluate further the β-dependence, starting from

$$\ln Z = -\frac{1}{2}\sum_n\sum_{\mathbf{p}}\ln\left[\beta^2\left(\omega_n^2+\mathbf{p}^2+m^2\right)\right].$$

We rewrite this in a form which enables us to do the summation over n. We abbreviate $\omega^2 = \mathbf{p}^2 + m^2$ and insert

$$\ln\left[(2\pi n)^2 + (\beta\omega)^2\right] = \int_1^{(\beta\omega)^2} \frac{\mathrm{d}(x^2)}{(2\pi n)^2 + x^2} + c$$

into the partition function, omitting the constant:

$$\begin{aligned}
\ln Z &= -\frac{1}{2}\sum_n \sum_{\mathbf{p}} \int_1^{(\beta\omega)^2} \frac{\mathrm{d}(x^2)}{(2\pi n)^2 + x^2} \\
&= -\frac{1}{2}\sum_{\mathbf{p}} \int_1^{(\beta\omega)^2} \frac{1}{2}\frac{\coth(x/2)}{x}\mathrm{d}\,(x^2) \\
&= -\frac{1}{2}\sum_{\mathbf{p}} \int_1^{\beta\omega} \coth(x/2)\mathrm{d}x \\
&= -\sum_{\mathbf{p}} \ln\sinh(\beta\omega/2)
\end{aligned}$$

where we have inserted $\sum_{n=-\infty}^{\infty} 1/(n^2 + x^2) = \pi/a \coth(\pi a)$[1] and $\int \coth(x)\mathrm{d}x = \ln\sinh(x)$ and we have omitted the β-independent lower integration boundary.

We continue by taking the continuum limit of the Fourier transform $\sum_{\mathbf{p}} \to V/(2\pi)^3 \int \mathrm{d}^3p$ and we insert $\ln\sinh(x) = 1/2 + x/2 + \ln(1 - \exp(-2x))$:

$$\ln Z = V\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[-\frac{1}{2} - \frac{\beta\omega}{2} - \ln\left(1 - e^{-\beta\omega}\right)\right].$$

The constant inside the square bracket is omitted because it is β-independent, the term proportionate to β is the (highly UV-divergent) zero-point energy which we need to subtract anyway because we want $P = 0$ in the limit $\beta \to \infty$. We consider the massless limit $\omega = |\mathbf{p}|$ to arrive at

$$\begin{aligned}
\ln Z &= V\int \frac{\mathrm{d}^3p}{(2\pi)^3}(4\pi)(-1)\int \mathrm{d}\omega\omega^2 \ln\left(1 - e^{-\beta\omega}\right) \\
&= V\frac{\pi^2}{90\beta^3}
\end{aligned}$$

from which we have $P = \beta^{-1}\partial_V \ln Z = \beta^{-4}\pi^2/90$.

[1] We use the residue theorem for this, the function $\pi i \coth(\pi i x)$ has residue one for $x \in \mathbb{Z}$. Therefore for $ia \notin \mathbb{Z}$ the function $\pi i \coth(\pi i x)/[(x + ia)(x - ia)]$ has residue $1/(x^2 + a^2)$ for $x \in \mathbb{Z}$. We consider the integration along the contour $C : z(\phi) = R\exp(i\phi)$ which vanishes as $R \to \infty$. From

$$0 = \lim_{R\to\infty}\int_C \pi i \coth(\pi i z)/[(z + ia)(z - ia)]\mathrm{d}z = \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2} + \pi i \coth(-\pi a)\frac{1}{2ia} + \pi i \coth(\pi a)\frac{1}{-2ia}$$

we get the result inserted above. To check $|\coth(R\exp i\phi)| \le c$ we insert

$$\left|\coth\left(Re^{i\phi}\right)\right|^2 = \left|\frac{e^{R(\cos\phi + i\sin\phi)} + e^{-R(\cos\phi + i\sin\phi)}}{e^{R(\cos\phi + i\sin\phi)} - e^{-R(\cos\phi + i\sin\phi)}}\right|^2 = \frac{1 + \frac{\cos(2R\sin\phi)}{\cosh(2R\cos(\phi))}}{1 - \frac{\cos(2R\sin\phi)}{\cosh(2R\cos(\phi))}}.$$

Although the function $\cos(2R\sin(\phi))/\cosh(2R\cos(\phi))$ is equal to one at $\phi = \pi/2 + n_1\pi, R = n_2\pi$, we can choose the sequence of radii R_n such that $\cos(2R_n\sin(\phi))/\cosh(2R_n\cos(\phi)) \le c' < 1$ for all n.

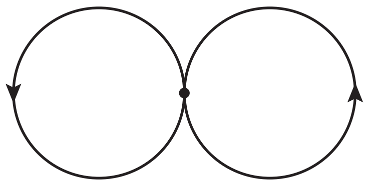

Figure 1: The two-loop vacuum bubble diagram of $\lambda\phi^4$ theory. Its symmetry factor is 3.

2. We condense the notation first, $\sum_n$ is used for the sum over n and $\mathbf{p}$, $\exp(i\omega_n\tau)$ should be understood as $\exp(i\omega_n\tau + i\mathbf{px})$ in consequence. Inserting the Fourier transforms we have therefore

$$\ln Z_1 = \frac{-\lambda\sum_{n_1,\dots,n_4}\frac{\beta^2}{V^2}\int \mathrm{d}\tau\int \mathrm{d}^3x\int\prod_k \mathrm{d}\phi_k e^{-\frac{1}{2}\beta^2 A_k|\phi_k|^2}\left(\prod_{i=1}^4 \phi_{n_i}\right)e^{i(\omega_{n_1}+\dots+\omega_{n_4})\tau}}{\prod_k \mathrm{d}\phi_k e^{-\frac{1}{2}\beta^2 A_k|\phi_k|^2}}.$$

with $A_k = \omega_k^2 + \mathbf{p}^2 + m^2$. The integral over ϕ_k vanishes by antisymmetry under $\phi_k \to -\phi_k$ unless the term is of the form $\phi_{r_1}\phi_{-r_1}\phi_{r_2}\phi_{-r_2} = |\phi_{r_1}|^2|\phi_{r_2}|^2$. In the four-fold sum every term $|\phi_{r_1}|^4$ shows up once, every term $|\phi_{r_1}|^2|\phi_{r_2}|^2$ with $r_1 \neq r_2$ shows up three times ($r_1 = -r_2$, $r_1 = -r_3$, $r_1 = -r_4$). We insert

$$\int \mathrm{d}x \exp^{-\frac{1}{2}ax^2}\mathrm{d}x = \sqrt{2\pi}a^{-\frac{1}{2}},$$
$$\int \mathrm{d}x \exp^{-\frac{1}{2}ax^2}x^2\mathrm{d}x = \sqrt{2\pi}a^{-\frac{3}{2}},$$
$$\int \mathrm{d}x \exp^{-\frac{1}{2}ax^2}x^4\mathrm{d}x = 3\sqrt{2\pi}a^{-\frac{5}{2}}$$

for every $\mathrm{d}\phi_m$. All the terms $\sqrt{2\pi}a^{-1/2}$ in the numerator are cancelled by equal terms in the denominator. We are left with a double sum

$$\ln Z_1 = -3\lambda\sum_{r_1,\mathbf{p}_1,r_2,\mathbf{p}_2}\frac{\beta^2}{V^2}\left(\beta^2\left[\omega_{r_1}^2 + \mathbf{p}_1^2 + m^2\right]\right)^{-1}\left(\beta^2\left[\omega_{r_2}^2 + \mathbf{p}_2^2 + m^2\right]\right)^{-1}\beta V$$

where the trailing βV arises from $\int \mathrm{d}^3x\mathrm{d}\tau\delta(\mathbf{x})\delta(\tau)$ and we have written the combinatorial factor 3 explicitly. We take the continuum limit of the Fourier transform as above:

$$\ln Z_1 = -3\lambda\beta V\left[\beta^{-1}\sum_n\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}\right]^2. \tag{1}$$

We recognise the expression above as the two-loop integral in figure 1 with $(\omega_n^2 + \mathbf{p}^2 + m^2)^{-1}$ as the propagator associated with an internal line.

3. We evaluate $\ln Z_1$ for $m = 0$, we have therefore $\omega = |\mathbf{p}|$. First we do the sum in the propagator:

$$\sum_n\frac{1}{\omega_n^2 + \omega^2} = \frac{1}{2\omega}\coth\left(\frac{\beta\omega}{2}\right).$$

We evaluate the integral over Fourier modes with a cutoff:

$$\int_{|\mathbf{p}|\leq\Lambda}\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega}\coth\left(\frac{\beta\omega}{2}\right) = \frac{1}{4\pi^2}\int_0^\Lambda\omega\coth\left(\frac{\beta\omega}{2}\right)$$
$$= \frac{1}{4\pi^2}\left[\frac{\pi^2}{3\beta^2} + \frac{\Lambda^2}{2} + \frac{2\Lambda\ln\left(1 - e^{-\beta\Lambda}\right)}{\beta} - \frac{2\,\mathrm{Li}_2\left(e^{-\beta\Lambda}\right)}{\beta^2}\right].$$

The second term in the expression above is β-independent and is therefore omitted, the other two Λ-dependent terms vanish in the limit $\Lambda \to \infty$. If we insert this back into (1), we arrive at

$$P_1 = \beta^{-1} \partial_V \ln Z_1 = -3\lambda \left(\frac{1}{12\beta^2} \right)^2 = -\frac{\lambda}{48} \beta^{-4}.$$

References

[Kap89] Joseph I. Kapusta. *Finite-temperature field theory.* Cambridge University Press, 1989.