ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Advanced Field Theory
Solution 10

Spring 2010
C. Anastasiou
T. Gehrmann

Exercise 10.1 The soliton of $\lambda\phi^4$ theory in $1+1$ dimensions

1. With
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \qquad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$
the constant solutions are $\phi(x) = \pm\sqrt{6}\,|m|\,/\sqrt{\lambda}$, therefore we shift the potential to be
$$V'(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{3}{2}\frac{|m|^4}{\lambda}.$$

2. The static soliton solutions are given by
$$\phi(x) = \frac{\sqrt{6}\,|m|}{\sqrt{\lambda}}\tanh\left(\frac{|m|}{\sqrt{2}}x\right).$$

3. We have
$$E = \int \mathrm{d}x\frac{1}{2}(\partial_x\phi)^2 + V'(\phi)$$
we insert
$$\partial_x\phi = \frac{\sqrt{3}\,|m|^2}{\sqrt{\lambda}}\cosh^{-2}\left(\frac{|m|}{\sqrt{2}}x\right), \qquad \frac{1}{2}(\partial_x\phi)^2 = \frac{3\,|m|^4}{2\lambda}\cosh^{-4}\left(\frac{|m|}{\sqrt{2}}x\right)$$
which we combine with
$$\begin{aligned} V'(\phi) &= \frac{-1}{2}\,|m|^2\,\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{3}{2}\frac{|m|^4}{\lambda} \\ &= \frac{3\,|m|^4}{2\lambda}\left(-2\tanh^2\left(\frac{|m|}{\sqrt{2}}x\right) + \tanh^4\left(\frac{|m|}{\sqrt{2}}x\right) + 1\right) \\ &= \frac{3\,|m|^4}{2\lambda}\cosh^{-4}\left(\frac{|m|}{\sqrt{2}}x\right) \end{aligned}$$
to have
$$E = \frac{3\,|m|^4}{2\lambda}\frac{\sqrt{2}}{|m|}\frac{8}{3} = \frac{4\sqrt{2}\,|m|^3}{\lambda}$$

4. We normalise
$$J^\mu = \frac{\sqrt{\lambda}}{2\sqrt{6}\,|m|}\epsilon^{\mu\nu}\partial_\nu\phi.$$
$\partial_\mu J^\mu$ vanishes due to the antisymmetry of ϵ, the possible values are given by
$$\int J^0 = \frac{\sqrt{\lambda}}{2\sqrt{6}\,|m|}\int \mathrm{d}x\partial_x\phi = \frac{\sqrt{\lambda}}{2\sqrt{6}\,|m|}\,\phi|_{-\infty}^{\infty} = \pm 1, 0$$
corresponding to the fact that there are two possible asymptotic values of $\phi(x)$.