

Exercise 6.1 Semileptonic Tau Decay: $\tau^+ \rightarrow \bar{\nu}_\tau \pi^+$

We cross $k \rightarrow -k$, $p \rightarrow -p$, adding an overall (-1) because we have crossed a fermion and a prefactor $1/2$ because we are now averaging over the spin of the incoming tau to arrive at

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}_{\ell^- \rightarrow \pi^- \nu_\ell}|^2 = 4G_F^2 f_\pi^2 (2(q \cdot p)(k \cdot p) - p^2(q \cdot k)).$$

We determine the scalar products from $k = p + q$, $k^2 = m_\ell^2$, $p^2 = m_\pi^2$ and $q^2 = 0$ as

$$p \cdot q = \frac{1}{2}(m_\ell^2 - m_\pi^2), \quad k \cdot q = \frac{1}{2}(m_\ell^2 - m_\pi^2), \quad k \cdot p = \frac{1}{2}(m_\ell^2 + m_\pi^2) \quad (1)$$

which we insert to have

$$\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = 2G_F^2 f_\pi^2 m_\ell^4 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right).$$

We combine this with the integrated two-particle phase space

$$\Pi_2 = \frac{1}{4\pi} \frac{1}{2} \left(1 - \frac{m_\pi^2}{m_\ell^2}\right)$$

to arrive at

$$\Gamma = \frac{1}{2m_\ell} \left(\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2\right) \Pi_2 = \frac{1}{8\pi} G_F^2 f_\pi^2 m_\ell^3 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right)^2.$$

Exercise 6.2 Semileptonic Tau Decay: $\tau^+ \rightarrow \bar{\nu}_\tau \rho^+$

In analogy to exercise sheet 4, the matrix element is the contraction of the hadronic current with the leptonic current

$$\mathcal{M} = 2G_F g_\rho \bar{v}(k) \gamma^\mu P_L v(q) \epsilon_{\mu,\lambda}(p) = 2G_F g_\rho \bar{v}(k) \not{\epsilon}_\lambda(p) P_L v(q).$$

We average over incoming spins and sum over outgoing spins and polarizations:

$$\frac{1}{2} \sum_{\text{spins, polarisations}} |\mathcal{M}|^2 = \langle |\mathcal{M}|^2 \rangle = 2G_F^2 g_\rho^2 \text{Tr} \left((\not{k} - m_\ell) \gamma^\mu P_L \not{q} \gamma^\nu \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_\rho^2} \right) \right)$$

where we have inserted the polarisation sum for the ρ polarisation vectors already. We expand this into

$$\langle |\mathcal{M}|^2 \rangle = 2G_F^2 g_\rho^2 \left(-\text{Tr} \left((\not{k} - m_\ell) \gamma^\mu P_L \not{q} \gamma_\mu \right) + \frac{1}{m_\rho^2} \text{Tr} \left((\not{k} - m_\ell) \not{p} P_L \not{q} \not{p} \right) \right).$$

We observe that the mass vanishes from the trace because the terms involving it have an odd number of Dirac matrices and that we can insert $P_L = 1/2$ inside the trace because the traces involving γ^5 vanish because there are no four linearly independent momenta in the process. We have therefore

$$\langle |\mathcal{M}|^2 \rangle = 2G_F^2 g_\rho^2 \left(-\frac{1}{2} \text{Tr} \left(\not{k} \gamma^\mu \not{q} \gamma_\mu \right) + \frac{1}{2m_\rho^2} \text{Tr} \left(\not{k} \not{p} \not{q} \not{p} \right) \right).$$

Now we use the contraction identity $\gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$ in the first trace and $\not{p}\not{q} = -\not{q}\not{p} + 2(p \cdot q)$ in the second one to reduce the traces to

$$\langle |\mathcal{M}|^2 \rangle = 2G_F^2 g_\rho^2 \left(\text{Tr}(\not{k}\not{q}) - \frac{p^2}{2m_\rho^2} \text{Tr}(\not{k}\not{q}) + \frac{p \cdot q}{m_\rho^2} \text{Tr}(\not{k}\not{p}) \right).$$

Now we insert $\text{Tr}(\gamma^\alpha \gamma^\beta) = 4g^{\alpha\beta}$ and $p^2 = m_\rho^2$ to have

$$\langle |\mathcal{M}|^2 \rangle = 2G_F^2 g_\rho^2 \left(2(k \cdot q) + \frac{4(p \cdot q)(k \cdot p)}{m_\rho^2} \right).$$

We insert equation (1) and expand to arrive at

$$\langle |\mathcal{M}|^2 \rangle = 2G_F^2 g_\rho^2 \left(m_\ell^2 - 2m_\rho^2 + \frac{m_\ell^4}{m_\rho^2} \right) = 2G_F^2 g_\rho^2 \frac{m_\ell^4}{m_\rho^2} \left(1 - \frac{m_\rho^2}{m_\ell^2} \right) \left(1 + 2\frac{m_\rho^2}{m_\ell^2} \right).$$

From this we get the partial width

$$\Gamma = \frac{1}{2m_\ell} \frac{1}{8\pi} \left(1 - \frac{m_\rho^2}{m_\ell^2} \right) \langle |\mathcal{M}|^2 \rangle = \frac{1}{8\pi} G_F^2 g_\rho^2 \frac{m_\ell^3}{m_\rho^2} \left(1 - \frac{m_\rho^2}{m_\ell^2} \right)^2 \left(1 + 2\frac{m_\rho^2}{m_\ell^2} \right).$$