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Advanced Field Theory
Assignment 2

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Exercise 2.1 Axial Anomaly in massive QED

Consider massive QED in 4 dimensions:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi$$

with $D_\mu = \partial_\mu + ieA_\mu$. Compute the divergence of the axial current, i.e. $\partial_\mu J^{\mu 5}$, with

$$J^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

To get a nonvanishing result even for vanishing mass, you have to define the axial current as

$$\underset{\delta\to 0}{\text{symm lim}} \left(\bar{\psi}\left(x+\frac{\delta}{2}\right)\gamma^\mu\gamma^5 \exp\left[-ie\int_{x-\delta/2}^{x+\delta/2} dz A(z)\right]\psi\left(x-\frac{\delta}{2}\right)\right)$$

where the symmetric limit means that the limiting procedure does not favor any direction, means

$$\underset{\delta\to 0}{\text{symm lim}}\left(\frac{\delta^\mu}{\delta^2}\right) = 0, \quad \underset{\delta\to 0}{\text{symm lim}}\left(\frac{\delta^\mu\delta^\nu}{\delta^2}\right) = \frac{g^{\mu\nu}}{4}.$$

Exercise 2.2 Fermion number nonconservation

1. In massless QED, the divergence of the axial current is given by the Adler-Bell-Jackiw anomaly:

$$\partial_\mu J^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

   Using this relation, show that:

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2}\int \mathrm{d}^4x\, \mathbf{E}\cdot\mathbf{B}$$

   where N_R and N_L are the numbers of right- and left-handed fermions, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively.

2. Show that the Hamiltonian for massless fermions, assuming $A_0 = 0$, is

$$H = \int \mathrm{d}^3x \left[\psi_R^\dagger\left(-i\sigma\cdot\mathbf{D}\right)\psi_R - \psi_L^\dagger\left(-i\sigma\cdot\mathbf{D}\right)\psi_L\right]$$

   where $\mathbf{D} = \nabla - ie\mathbf{A}$.
   Here you have to use a chiral basis for the gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

   with $\sigma^\mu = (1,\sigma)$, $\bar{\sigma}^\mu = (1,-\sigma)$ and σ^i the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Consider a background field given by $A^\mu = (0, 0, Bx^1, A)$, where B is constant and A is constant in space and varying adiabatically in time. In order to diagonalise the Hamiltonian of 2, one has to solve the eigenvalue equation $-i\sigma \cdot \mathbf{D}\psi_R = E\psi_R$, which can be achieved by writing:

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i\left(k_2 x^2 + k_3 x^3\right)}$$

Derive the differential equations for ϕ_1 and ϕ_2, show that they obey the equation of a simple harmonic oscillator and use this observation to find the single-particle spectrum of the Hamiltonian.

4. Now let's assume that the system is set up in a cubic box of length L, with periodic boundary conditions. Then the momenta will be quantised:

$$k_i = \frac{2\pi n_i}{L} \qquad\qquad n_i \in \mathbb{Z}.$$

Show that the condition that the center of the oscillation is inside the box leads to the condition

$$k_2 < eBL$$

so that each energy level has a degeneracy of

$$\frac{eL^2 B}{2\pi}$$

5. Finally consider changing the background A adiabatically by an amount

$$\Delta A^1 = \frac{2\pi}{eL}$$

Show that the vacuum loses right-handed fermions.

Similarly, compute the eigenvalues for the left-handed fermions and show that under the same change ΔA^1 the vacuum gains the same number of left-handed fermions. Show that these numbers agree with the global nonconservation law of part 1.