Solid State Theory
Exercise 9

Landau Fermi liquid theory

Exercise 9.1 Uniaxial Compressibility

We consider a system of electrons upon which an uniaxial pressure in z-direction acts. Assume that this pressure causes a deformation of the Fermi surface $k \equiv k_F^0$ of the form

$$k_F(\phi, \theta) = k_F^0 + \gamma \frac{1}{k_F^0}\left[3k_z^2 - (k_F^0)^2\right] = k_F^0 + \gamma k_F^0[3\cos^2\theta - 1], \tag{1}$$

where $\gamma = (P_z - P_0)/P_0$ is the anisotropy of the applied pressure.

a) Show that for small $\gamma \ll 1$, the deformed Fermi surface $k_F(\phi, \theta)$ encloses the same volume as the non-deformed one, k_F^0, where terms of order $\mathcal{O}(\gamma^2)$ can be neglected.

b) The deformation of the Fermi surface effects a change in the distribution function of the electrons. Using Landau's Fermi Liquid theory, calculate the uniaxial compressibility

$$\kappa_u = \frac{1}{V}\frac{\partial^2 E}{\partial P_z^2}, \tag{2}$$

which is caused by the deformation given in Eq. (1) (E denotes the Landau energy functional).

c) What is the stability condition for the deformation given in Eq. (1) of a Fermi liquid?

Exercise 9.2 Pomeranchuk instability

It can be shown in general [1] that the Fermi liquid is stable against an arbitrary deformation

$$k_F(\phi, \theta) = k_F^0 + u_\sigma(\phi, \theta) \tag{3}$$

of the Fermi surface if

$$F_l^s > -(2l+1), \tag{4}$$

$$F_l^a > -(2l+1). \tag{5}$$

Verify this result by considering Landau's energy functional and expanding the displacement $u_\sigma(\phi, \theta)$ in terms of spherical harmonics.

Exercise 9.3 Polarization of a neutral Fermi liquid

Consider a system of neutral spin-1/2 particles each carrying a magnetic moment $\vec{\mu} = \frac{\mu}{2}\vec{\sigma}$. An electric field $\vec{E}$ couples to the atoms by the relativistic spin-orbit interaction

$$H_{SO} = \frac{\mu}{2}\left(\frac{\vec{v}}{c} \times \vec{E}\right) \cdot \vec{\sigma} \tag{6}$$

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli spin matrices. In the following we want to calculate the linear response function χ for the uniform polarization

$$\vec{P} = \chi \vec{E}. \tag{7}$$

In the presence of spin-orbit interaction we have to consider a more general situation of a distribution of quasiparticles with variable spin quantization axis. In such a case we must treat the quasiparticle distribution function and the energy as a 2×2 matrix, $(\hat{n}_{\vec{p}})_{\alpha\beta}$ and $(\hat{\epsilon}_{\vec{p}})_{\alpha\beta}$, respectively. Furthermore, we require f to be a scalar under spin rotations. In this case f must be of the form

$$\hat{f}_{\alpha\beta,\alpha'\beta'}(\vec{p},\vec{p}') = f^s(\vec{p},\vec{p}')\delta_{\alpha\beta}\delta_{\alpha'\beta'} + f^a(\vec{p},\vec{p}')\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\alpha'\beta'} \tag{8}$$

a) Expand $\hat{n}_{\vec{p}}$, $\hat{\epsilon}_{\vec{p}}$, and $\hat{f}_{\vec{\sigma}\vec{\sigma}'}(\vec{p},\vec{p}')$ in terms of the unit matrix $\sigma^0 = \mathbf{1}$ and the Pauli spin matrices $\sigma^1 = \sigma^x$, $\sigma^2 = \sigma^y$, $\sigma^3 = \sigma^z$ and find Landau's energy functional E.

b) Assume that the electric field is directed along the z direction. Show that the polarization of such a system is given by

$$P_z = \frac{\partial E}{\partial E_z} = \frac{\mu}{m^* c} \sum_{\vec{p}} \left(p_y \delta n^1_{\vec{p}} - p_x \delta n^2_{\vec{p}} \right). \tag{9}$$

Here, $\delta n^i_{\vec{p}} = \frac{1}{2}\mathrm{tr}\left[\delta \hat{n}_{\vec{p}} \sigma^i\right]$ and $\delta \hat{n}_{\vec{p}}$ is the deviation from the equilibrium ($E_z = 0$) distribution function.

c) The application of the electric field changes the quasiparticle energy in linear response according to

$$\delta\tilde{\epsilon}^i_{\vec{p}} = \delta\epsilon^i_{\vec{p}} + \frac{2}{V}\sum_{\vec{p}'} f^{ii}(\vec{p},\vec{p}')\delta n^i_{\vec{p}'} \quad \text{with} \quad \delta n^i_{\vec{p}} = \frac{\partial n_0}{\partial \epsilon}\delta\tilde{\epsilon}^i_{\vec{p}} = -\delta(\epsilon^0_{\vec{p}} - \epsilon_F)\delta\tilde{\epsilon}^i_{\vec{p}}. \tag{10}$$

Use the ansatz $\delta\tilde{\epsilon}^i_{\vec{p}} = \alpha\delta\epsilon^i_{\vec{p}}$ and show that $\alpha = 1/(1 + F^a_1/3)$ to find $\delta n^i_{\vec{p}}$ and $\delta\tilde{\epsilon}^i_{\vec{p}}$.

d) Compute χ according to Eq. (7).

References

[1] Pomeranchuk, Ia., *Sov. Phys. JETP* **8**, 361 (1958).