# Quantum Mechanics II
# Exercise 12



## Exercise 12.1 Many-Body Perturbation Theory

We begin with a definition of the field Hamiltonian $\hat{\mathcal{H}}_{\mathcal{F}}$ of a generic $N$-particle system,

$$\hat{\mathcal{H}}_{\mathcal{F}} = \hat{\mathcal{H}}_{0,\mathcal{F}} + \frac{1}{2}\hat{\mathcal{V}}_{\mathcal{F}} \;, \tag{1}$$

where $\hat{\mathcal{H}}_{0,\mathcal{F}}$ is the free particle Hamiltonian,

$$\hat{\mathcal{H}}_{0,\mathcal{F}} = \int \mathrm{d}^3k\, \frac{\hbar^2\,|\mathbf{k}|^2}{2m}\, \hat{b}^\dagger(\mathbf{k},t)\, \hat{b}(\mathbf{k},t) \;, \tag{2}$$

and $\hat{\mathcal{V}}_{\mathcal{F}}$ represents the two-particle interaction,

$$\hat{\mathcal{V}}_{\mathcal{F}} = \int \frac{\mathrm{d}^3k_1\,\mathrm{d}^3k_2\,\mathrm{d}^3q}{\left(\sqrt{2\pi}\right)^3}\, \tilde{V}(\mathbf{q})\, \hat{b}^\dagger(\mathbf{k}_1,t)\, \hat{b}^\dagger(\mathbf{k}_2,t)\, \hat{b}(\mathbf{k}_2-\mathbf{q},t)\, \hat{b}(\mathbf{k}_1+\mathbf{q},t) \;. \tag{3}$$

Here, $\tilde{V}(\mathbf{q})$ is the Fourier transform of the interaction potential and the operators $\hat{b}^\dagger(\mathbf{p},t)$ and $\hat{b}(\mathbf{p},t)$ create respectively destroy a particle with momentum $\mathbf{p}$ at time $t$.

a) Show that the $N$ particle state, defined by

$$|\,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\ldots,\mathbf{p}_N\,\rangle \equiv \hat{b}^\dagger(\mathbf{p}_1,t)\,\hat{b}^\dagger(\mathbf{p}_2,t)\,\hat{b}^\dagger(\mathbf{p}_3,t)\,\ldots\,\hat{b}^\dagger(\mathbf{p}_N,t)\,|\,\mathbf{0}\,\rangle \;, \tag{4}$$

is an eigenstate of the free field Hamiltonian $\hat{\mathcal{H}}_{0,\mathcal{F}}$, i.e.

$$\hat{\mathcal{H}}_{0,\mathcal{F}}\,|\,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\ldots,\mathbf{p}_N\,\rangle = E^{(0)}\,|\,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\ldots,\mathbf{p}_N\,\rangle \;, \tag{5}$$

with the non-interacting energy eigenvalue $E^{(0)}$, given by

$$E^{(0)} = \sum_{i=1}^{N} \frac{\hbar^2|\mathbf{p}_i|^2}{2m} \;. \tag{6}$$

Note that $E^{(0)}$ represents the energy of the non-interacting system in the the many-body state $|\,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\ldots,\mathbf{p}_N\,\rangle$ and should not be confused with the ground state energy of the system.

Show the above statement for a small number of particles ($N=2$ and $N=3$) first. Then, generalize to arbitrary particle numbers.

b) We now study the properties of the interaction part $\hat{\mathcal{V}}_{\mathcal{F}}$ of the full Hamiltonian (1).

Show the following relations:

(i) $$\langle\,\mathbf{k}\,|\,\mathbf{p}\,\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{7}$$

(ii) $$\langle\,\mathbf{k}_1,\mathbf{k}_2\,|\,\mathbf{p}_1,\mathbf{p}_2\,\rangle = \delta^{(3)}(\mathbf{k}_1-\mathbf{p}_1)\delta^{(3)}(\mathbf{k}_2-\mathbf{p}_2) \pm \delta^{(3)}(\mathbf{k}_1-\mathbf{p}_2)\delta^{(3)}(\mathbf{k}_2-\mathbf{p}_1) \tag{8}$$

(iii) $$\hat{\mathcal{V}}_{\mathcal{F}}\,|\,\mathbf{p}_1\,\rangle = 0 \tag{9}$$

(iv) $$\hat{\mathcal{V}}_{\mathcal{F}}\,|\,\mathbf{p}_1,\mathbf{p}_2\,\rangle = \int \frac{\mathrm{d}^3q}{(2\pi)^{3/2}}\tilde{V}(\mathbf{q})\Big(|\,\mathbf{p}_1-\mathbf{q}\,,\mathbf{p}_2+\mathbf{q}\rangle \pm |\,\mathbf{p}_2-\mathbf{q}\,,\mathbf{p}_1+\mathbf{q}\rangle\Big) \tag{10}$$

Here, the plus (minus) sign corresponds to the description of bosons (fermions).

*Hint:*
First, express the states in terms of creation operators acting on the vacuum, as was defined in Eq. (4). Then, make use of the well-known commutation (anti-commutation) relations of the creation and annihilation operators in order to generate terms in which an annihilation operator acts on the vacuum state. The commutation and anti-commutation relations read

$$\begin{aligned}
\textit{bosons:} \quad & [\,\hat{b}(\mathbf{p},t), \hat{b}^\dagger(\mathbf{p}',t)\,]_- \equiv \hat{b}(\mathbf{p},t)\,\hat{b}^\dagger(\mathbf{p}',t) - \hat{b}^\dagger(\mathbf{p}',t)\,\hat{b}(\mathbf{p},t) = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\ , \\
& [\,\hat{b}^\dagger(\mathbf{p},t), \hat{b}^\dagger(\mathbf{p}',t)\,]_- = [\,\hat{b}(\mathbf{p},t), \hat{b}(\mathbf{p}',t)\,]_- = 0\ ,
\end{aligned}$$

$$\begin{aligned}
\textit{fermions:} \quad & [\,\hat{b}(\mathbf{p},t), \hat{b}^\dagger(\mathbf{p}',t)\,]_+ \equiv \hat{b}(\mathbf{p},t)\,\hat{b}^\dagger(\mathbf{p}',t) + \hat{b}^\dagger(\mathbf{p}',t)\,\hat{b}(\mathbf{p},t) = \delta^{(3)}(\mathbf{p}-\mathbf{p}')\ , \\
& [\,\hat{b}^\dagger(\mathbf{p},t), \hat{b}^\dagger(\mathbf{p}',t)\,]_+ = [\,\hat{b}(\mathbf{p},t), \hat{b}(\mathbf{p}',t)\,]_+ = 0\ .
\end{aligned}$$

### Exercise 12.2 Transition Amplitudes

The first step towards a perturbative expansion of observables of an interacting many-particle system is to calculate transition amplitudes.
In this exercise we again assume the system to be described by the Hamiltonian of Exercise 12.1, Eq. (1).
Compute the "three-particle-to-three-particle" transition amplitude:

$$\langle\, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \,|\, \frac{1}{2}\hat{\mathcal{V}}_{\mathcal{F}} \,|\, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \,\rangle \tag{11}$$

*Hint:*
Again, take advantage of the commutation relations of the creation and annihilation operators and the action of the annihilation operator on the vacuum state. Treating the two cases of fermionic and bosonic operators independently might help not to get lost with $\pm$ and $\mp$ signs.