Exercise 4.1 Formalism of time-dependent perturbation theory

a) Prove that

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1)H(t_2)\cdots H(t_n) =$$
$$= \frac{1}{n!} \iiint_{t_0}^{t} dt_1 dt_2 \cdots dt_\eta T\left\{H(t_1)H(t_2)\cdots H(t_n)\right\} \quad (1)$$

where $T\{\cdots\}$ denotes time-ordering.

b) Prove that

$$|\Psi, t\rangle = T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')\right]\right\}|\Psi, t_0\rangle \quad (2)$$

is a solution of the Schrödinger equation.

c) Prove that

$$|\Psi, t\rangle = T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')\right]\right\}|\Psi, t_0\rangle \quad (3)$$

reduces to

$$|\Psi, t\rangle = \exp\left[-\frac{i}{\hbar}H(t-t_0)\right]|\Psi, t_0\rangle \quad (4)$$

for a closed system, i.e. $\partial_t H = 0$.

Exercise 4.2 Hydrogen atom in an electric field

Consider a hydrogen atom in its ground state (n=1), which, beyond time $t = 0$, is subject to a spatially uniform time-dependent electric field pointing in z-direction, $\mathcal{E}_0 e^{-t/\tau}\hat{z}$.

a) What are the allowed transitions for the hydrogen atom?

b) Treating the electric field as a perturbation, calculate to first order the probability of finding the atom in the first excited state (n=2)