ETH
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Quantum Information Theory
Serie 2

FS 09
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Exercise 2.1 Min-Entropy

The classical min-entropy of a probability distribution P_X over $\mathcal{X}$ is defined as

$$H_{\min}(X) = \min_{x \in \mathcal{X}} h_P(x), \tag{1}$$

where the information content of an event $\{X = x\}$ is given by $h_P(x) = -\log P_X(x)$. Show that $H_{\min}(X) \leq \log |\mathcal{X}|$ for any distribution P_X over $\mathcal{X}$.

Exercise 2.2 Min-Entropy in the i.i.d. limit

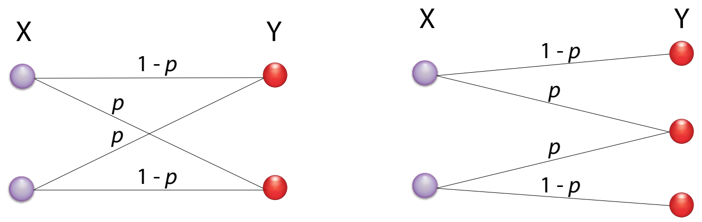
Let us introduce the "smoothed" min-entropy of a random variable X over $\mathcal{X}$ as

$$H^{\epsilon}_{\min}(X) = \max_{\substack{Q_X \\ \delta(Q_X, P_X) < \epsilon}} \min_{x \in \mathcal{X}} h_Q(x), \tag{2}$$

where $h_Q(x) = -\log Q_X(x)$ and the maximum is over all probability distributions Q_X that are ϵ-close to P_X. Further, we define an i.i.d. random variable $\vec{X} = \{X_1, X_2, \ldots, X_n\}$ on $\mathcal{X}^{\times n}$ with $P_{\vec{X}}(\vec{x}) = \prod_{i=1}^{n} P_X(x_i)$. Use the weak law of large numbers to show that the "smoothed" min-entropy converges to the Shannon entropy $H(X)$ in the i.i.d. limit:

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} H^{\epsilon}_{\min}(\vec{X}) \geq H(X). \tag{3}$$

Exercise 2.3 Channel capacity



(a) Binary Symmetric Channel (b) Symmetric Erasure Channel

The asymptotic channel capacity is given by

$$C = \max_{P_X} I(X : Y).$$

a) Calculate the asymptotic capacities of the two channels depicted above.

b) Can we transmit a message error-free and with a finite amount of channel uses in any of the two channels ($p > 0$)?

*c) Show that feedback does not improve the channel capacity.