Quantum Field Theory II, Exercise Set 9.

FS 08/09

1. Ghost Feynman Diagrams and Unitarity

We consider $q\bar{q} \to q\bar{q}$ forward scattering at one loop order. We shall work in the *Feynman gauge* ($\alpha = 1$, including FP-ghost diagrams).

a) Draw all Feynman diagrams contributing to this process at one-loop order. Draw the three tree-level diagrams for the process $q\bar{q} \to g^a g^b$ and write down their amplitudes. Denote the sum of these amplitudes by T^{ab}, where a and b are color indices. Draw the tree-level Feynman diagram for the process $q\bar{q} \to c^{a*}c^b$, compute its amplitude, which we denote by G^{ab}.

Recall from chapter 11, QFT1, that the scattering amplitude (or invariant matrix element) $\mathrm{T}_{fi} \equiv \langle f|\mathrm{T}|i\rangle$ is related to the scattering matrix by

$$S_{fi} = \delta_{fi} + \mathrm{i}(2\pi)^4\delta(p_i - p_f)\mathrm{T}_{fi}\,. \tag{1}$$

In this exercise we have $|i\rangle = |q\bar{q}\rangle$ and $|f\rangle = |q\bar{q}\rangle$.

b) Show that the unitarity of the scattering matrix implies

$$\mathrm{Im}\,\mathrm{T}_{fi} = \frac{1}{2}\sum_n \mathrm{T}_{fn}\mathrm{T}^*_{in}(2\pi)^4\delta(p_i - p_n)\,, \tag{2}$$

where the sum is over intermediate states, e.g., $|n\rangle = |gg\rangle$.

The two-gluon amplitudes must have the form

$$T^{ab} = T^{ab}_{\mu_1\nu_1}\varepsilon_1^{\mu_1}(k_1,\lambda_1)\varepsilon_2^{\nu_1}(k_2,\lambda_2)\,, \tag{3}$$

thus equation (2) becomes

$$\mathrm{Im}\,\mathrm{T}^{ab}_{fi} = \frac{1}{2}\int \mathrm{d}\Pi_2 \sum_{\lambda_1,\,\lambda_2=1,2} T^{ab}_{\mu_1\nu_1}\varepsilon_1^{\mu_1}(k_1,\lambda_1)\varepsilon_2^{\nu_1}(k_2,\lambda_2){T^{ab}_{\mu_2\nu_2}}^*\varepsilon_1^{\mu_2}(k_1,\lambda_1)\varepsilon_2^{\nu_2}(k_2,\lambda_2)\,, \tag{4}$$

where $\mathrm{d}\Pi_2$ is the two massless particle phase space measure. Note that we have chosen real polarisation vectors.

The goal of this exercise is to verify equation (4) explicitely. The imaginary part of the scattering amplitude T_{fi} in equation (4) can be calculated using the *Cutkosky* rule. In this simple case it states that the imaginary part is obtained by replacing the propagators in the intermediate states by their imaginary parts and multiplying them by the on-shell scattering amplitudes T^{ab} and T^{ab*}.

c) Show the equation (4) can be written as

$$\int \mathrm{d}\Pi_2\left[\frac{1}{2}T^{ab}_{\mu_1\nu_1}{T^{ab}_{\mu_2\nu_2}}^* g^{\mu_1\mu_2}g^{\nu_1\nu_2} - G^{ab}G^{ab*}\right] = \frac{1}{2}\int \mathrm{d}\Pi_2 T^{ab}_{\mu_1\nu_1}{T^{ab}_{\mu_2\nu_2}}^* P^{\mu_1\mu_2}(k_1)P^{\nu_1\nu_2}(k_2)\,, \tag{5}$$

where

$$
\begin{aligned}
P^{\mu_1\mu_2}(k_1) &:= \sum_{\lambda=1}^{2} \varepsilon_1^{\mu_1}(k_1,\lambda)\varepsilon_1^{\mu_2}(k_1,\lambda)\,, \\
P^{\nu_1\nu_2}(k_2) &:= \sum_{\lambda=1}^{2} \varepsilon_2^{\nu_1}(k_2,\lambda)\varepsilon_2^{\nu_2}(k_2,\lambda)\,,
\end{aligned} \tag{6}
$$

are polarisation sums. Where does the factor $\frac{1}{2}$ on the l.h.s. come from? Why is there a minus sign in front of the ghost terms?

Next we attempt to compute the polarisation sums in equation (6): The gauge bosons being massless have two physical polarisations. The three vectors k_μ, $\varepsilon_\mu(k,1)$, $\varepsilon_\mu(k,2)$ do not span he four dimensinal space. Thus we add a fourth vector η_μ such that

$$
\eta \cdot \varepsilon(k,\lambda) = 0 \quad \lambda = 1,2\,, \tag{7}
$$

and such that

$$
\varepsilon(k,1)\cdot\varepsilon(k,2) = 0\,, \quad k\cdot\varepsilon(k,\lambda) = 0\,, \quad \varepsilon^2(k,\lambda) = -1\,, \quad k\cdot\eta \neq 0\,. \tag{8}
$$

d) Show that

$$
P_{\mu\nu}(k) = -g_{\mu\nu} + \frac{(k\cdot\eta)(k_\mu\eta_\nu + k_\nu\eta_\mu) - \eta^2 k_\mu k_\nu}{(k\cdot\eta)^2}\,. \tag{9}
$$

Use that $\sum_{\lambda=0}^{3}\varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda) = -g^{\mu\nu}$. We may choose η light-like in the following.

e) Recall from QED that gauge invariance implies $k^{\mu_1}T_{\mu_1\nu_1} = 0$ (Ward identity), for any QED process involving external photons of momentum k. Verify that this does not hold in QCD by showing that

$$
\begin{aligned}
k_1^{\mu_1}T^{ab}_{\mu_1\nu_1} &= -\mathrm{i}G^{ab}k_{2\,\nu_1}\,, \\
T^{ab}_{\mu_1\nu_1}k_2^{\nu_1} &= -\mathrm{i}G^{ab}k_{1\,\mu_1}\,.
\end{aligned} \tag{10}
$$

It then follows that $k_1^\mu T^{ab}_{\mu\nu}k_2^\nu = 0$. This is a generalised Ward- or *Slavnov-Taylor* identity.

Hint: Use $(\not{p}_1 + m)\not{k}_1 u(p_1) = [2p_1\cdot k_1 - \not{k}_1(\not{p}_1 - m)]u(p_1) = 2p_1\cdot k_1 u(p_1)$ *and simlilar equations (Dirac- algebra and equation).*

f) Verify the unitarity condition, i.e., equation (5).

Hint: Use equations (10) *to replace* T^{ab} *by* G^{ab}*, this way all dependences on* η *will drop out as well.*