Quantum Field Theory II, Exercise Set ♯ 2.

FS 08/09 *Due: 11.03.09*

1. Charge conjugation

Let ψ be a solution of the Dirac equation

$$(\partial\!\!\!/ - \mathrm{i}eA\!\!\!/)\psi = 0\,. \tag{1}$$

The charge conjugate spinor ψ^C is defined as $\psi^C := \mathcal{C}\gamma^0(\psi^*)^T$, where $\mathcal{C} := \mathrm{i}\gamma^2\gamma^0$ is the charge conjugation operator in the chiral representation of the gamma matrices (see exercise 2) and ψ^* is the hermitian conjugate of the field ψ.

a) Show that if ψ satisfies (1), then for its charge conjugate we have

$$(\partial\!\!\!/ + \mathrm{i}eA\!\!\!/)\psi^C = 0\,.$$

b) Show that the charge conjugate of a right-handed field, $(\psi_R)^C$, is a left-handed field. More precisely $(\psi_R)^C = (\psi^C)_L$.

2. Majorana condition

The Dirac Lagrangian is given by

$$\mathcal{L} = \frac{\mathrm{i}}{2}(\overline{\Psi}\gamma_\mu\partial^\mu\Psi - \partial^\mu\overline{\Psi}\gamma_\mu\Psi) - m\overline{\Psi}\Psi\,.$$

We will work in the chiral representation of the Dirac algebra, i.e.,

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \hat{\sigma}_\mu & 0 \end{pmatrix}\,, \quad \text{or} \quad \gamma_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}\,, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\,,$$

and write a Dirac spinor in bispinor notation $\Psi = \begin{pmatrix} \varphi_\alpha \\ \chi^{\dot\beta} \end{pmatrix}$.

a) Find the Dirac equation in the bispinor formalism, as well as the Lagrange density.

A neutral particle is described by the above formalism by imposing the *Majorana condition*:

$$\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\overline{\varphi}_{\dot\beta}\,, \quad \varphi_\alpha = \varepsilon_{\alpha\beta}\overline{\chi}^\beta\,.$$

b) Show that the Dirac equation and the Majorana condition imply the Majorana equation

$$\mathrm{i}\hat{\sigma}_\mu^{\dot\alpha\beta}\partial^\mu\varphi_\beta = m\varepsilon^{\dot\alpha\dot\beta}\overline{\varphi}_{\dot\beta}\,.$$

Show that the corresponding Lagrange density vanishes.

Hint: Show that $\hat{\sigma}_\mu = \varepsilon^{\mathrm{T}}\sigma_\mu^{\mathrm{T}}\varepsilon$.

c) The way out is to think of φ_α and $\chi^{\dot\beta}$ as Grassmann variables. Show that the Euler-Lagrange equations of the Lagrange density

$$\mathcal{L} = \frac{\mathrm{i}}{2}(\overline{\varphi}_{\dot\alpha}\hat\sigma_\mu^{\dot\alpha\beta}\partial^\mu\varphi_\beta - \partial^\mu\overline{\varphi}_{\dot\beta}\hat\sigma_\mu^{\dot\beta\alpha}\varphi_\alpha) + \frac{m}{2}(\varphi_\alpha\varepsilon^{\alpha\beta}\varphi_\beta - \overline{\varphi}_{\dot\alpha}\varepsilon^{\dot\alpha\dot\beta}\overline{\varphi}_{\dot\beta}), \tag{2}$$

are the Majorana equation and its adjoint version. A mass term as in (2) is called a Majorana mass.

3. Abelian Higgs mechanism

Consider the theory of a complex scalar field φ described by the Hamilton functional

$$H(\pi, \varphi, \mathbf{E}, \mathbf{A}) = \frac{1}{2}\int d\mathbf{x}\left\{\pi^2 + |\nabla^A\varphi|^2 + V(\varphi) + \mathbf{E}^2 - (\nabla\times\mathbf{A})^2\right\},$$

where $\nabla_i^A = \partial_i - \mathrm{i}qA_i$, and the potential of the scalar field is

$$V(\varphi) = \frac{\lambda}{2}|\varphi|^4 - \mu^2|\varphi|^2, \quad \lambda, \mu^2 > 0.$$

a) Minimize the potential $V(\varphi)$. Note that the condition for the potential to be minimized fixes only the modulus of the field φ_0, but not the phase. Since all the minima are equivalent, one can choose this phase to be zero.

Consider now small oscillations $\chi(x)$ around the minimun φ_0, i.e. write the field $\varphi(x)$ as $\varphi(x) = \varphi_0 + \chi(x)$. Convince yourself that these small perturbations can be decomposed into a 'radial' component $\delta\rho(x)$ and an 'angular' component $\sigma(x)$, so that

$$\varphi(x) = \varphi_0 + \delta\rho(x) + \mathrm{i}\sigma(x). \tag{3}$$

b) Expand the kinetic term $|\nabla^A\varphi|^2$ (you can neglect terms containing more than three fields). Identify the mass of $\mathbf{A}$. Also make sure that the real fields $\delta\rho$ and σ are canonically normalized (look at their kinetic term). If not, define new fields $\delta\rho'$ and σ' that are.

c) Expand the potential $V(\varphi)$ around the minimum by means of (3). What are the masses of the (canonically normalized) radial and angular fluctuations? Looking at the three- and four-fields terms, which are the possible interactions?